

CLASSIFICATION OF NEW REALS

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joint work with **Bohuslav Balcar**

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NEW REALS

Phrase 'Forcing notion *adds a new real*' means that for any generic filter G over V , the generic extension $V[G]$ contains a new subset $\sigma \subset \omega$

Hence $V[G]$ contains a function $\rho : \omega \rightarrow \omega$ which does not belong to groundmodel V .

It is quite common in set theory that under the term 'real' we mean subset of ω . Hence elements of Cantor space $\mathcal{C} = {}^\omega\{0, 1\}$ are reals as well as the function from ω to ω , i.e. elements from Baire space \mathcal{N} are called reals.

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Let M denote an extension of V .

- $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^\omega \cap V$ both $X \cap Y$ and $Y - X$ are infinite.
- A function $f \in M$, $f \in \omega^\omega$, is a *dominating real* over V if for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.
- A function $h \in \omega^\omega$ in the extension is said to be an *unbounded real* over V if for all $f \in \omega^\omega \cap V$ the set $\{n \in \omega : h(n) > f(n)\}$ is infinite.
- A function $h \in \omega^\omega$ in the extension is said to be an *eventually different real* over V if for all $f \in \omega^\omega \cap V$ the set $\{n \in \omega : h(n) = f(n)\}$ is finite.
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COHEN REAL

Cohen forcing. *Cohen forcing* is countable atomless ordering and is equivalent to any of the following set

- $Seq = \bigcup \{ {}^n \omega : n < \omega \},$
- $Seq_2 = \bigcup \{ {}^n 2 : n < \omega \},$
- $Fn(\omega, 2) = \{ f; f : D \rightarrow \{0, 1\}, D \in [\omega]^{<\omega} \},$

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- adds a new real,
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- does not add an eventually different real, hence cannot add dominating reals.

RANDOM REAL

- $(\text{BOREL}(2^\omega) - \text{Null}, \subseteq)$ is **Random forcing**. The ordering is not separative, its separative quotient is
- $(\text{BOREL}(2^\omega)/\text{Null}, \subseteq)$. This is *ccc* complete atomless Boolean algebra that carries strictly positive σ -additive measure, $m[U] = m(U)$, for each $U \in \text{BOREL}(2^\omega)$.

fact Any measure algebra satisfies *ccc*.

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Theorem

- (i) In *Random* extension are groundmodel reals *meager*.
- (ii) In *Cohen* extension are groundmodel reals *negligible*.

DOMINATING REAL

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HECHLER REAL

Hechler forcing

is a set $H_0 = \{ \langle s, f \rangle : s \in {}^{<\omega}\omega, f : \omega \rightarrow \omega, s \subset f \}$,

with a partial ordering

$\langle s, f \rangle \leq \langle t, g \rangle$ if and only if $t \subseteq s$ & $(\forall n \in \omega) f(n) \geq g(n)$.

adds dominating real

Hence it also adds eventually different, unbounded, and independent real.

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