CLASSIFICATION OF NEW REALS

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joint work with Bohuslav Balcar

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New Reals

Phrase ’Forcing notion adds a new real’ means that for any generic filter $G$ over $V$, the generic extension $V[G]$ contains a new subset $\sigma \subseteq \omega$

Hence $V[G]$ contains a function $\rho : \omega \to \omega$ which does not belong to groundmodel $V$.

It is quite common in set theory that under the term ’real’ we mean subset of $\omega$. Hence elements of Cantor space $C = \omega \{0, 1\}$ are reals as well as the function from $\omega$ to $\omega$, i.e. elements from Baire space $N$ are called reals.
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Let $M$ denote an extension of $V$.

- $X \subseteq \omega$ in the extension is said to be an independent (or splitting) real over $V$ if for all $Y \in [\omega]^{\omega} \cap V$ both $X \cap Y$ and $Y - X$ are infinite.
- A function $f \in M$, $f \in \omega^{\omega}$, is a dominating real over $V$ if for all $g \in \omega^{\omega} \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.
- A function $h \in \omega^{\omega}$ in the extension is said to be an unbounded real over $V$ if for all $f \in \omega^{\omega} \cap V$ the set $\{n \in \omega : h(n) > f(n)\}$ is infinite.
- A function $h \in \omega^{\omega}$ in the extension is said to be an eventually different real over $V$ if for all $f \in \omega^{\omega} \cap V$ the set $\{n \in \omega : h(n) = f(n)\}$ is finite.
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Each dominating real is eventually different.
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Cohen forcing. Cohen forcing is countable atomless ordering and is equivalent to any of the following set

- $\text{Seq} = \bigcup \{^n \omega : n < \omega \}$,
- $\text{Seq}_2 = \bigcup \{^n 2 : n < \omega \}$,
- $\text{Fn}(\omega, 2) = \{f; f : D \to \{0, 1\}, D \in [\omega]^{<\omega}\}$,

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- adds a new real,
- adds a splitting set,
- adds unbounded real,
- does not add an eventually different real, hence cannot add dominating reals.
• \((\text{Borel}(2^\omega) - \text{Null}, \subseteq)\) is Random forcing. The ordering is not separative, its separative quotient is

• \((\text{Borel}(2^\omega)/\text{Null}, \subseteq)\). This is ccc complete atomless Boolean algebra that carries strictly positive \(\sigma\)-additive measure, \(m[U] = m(U)\), for each \(U \in \text{Borel}(2^\omega)\).

**fact** Any measure algebra satisfies ccc.
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Theorem
(i) In Random extension are groundmodel reals meager.
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Theorem
(i) In Random extension are groundmodel reals meager.
(ii) In Cohen extension are groundmodel reals negligible.
A function $f \in M, f \in \omega^\omega$, is a *dominating real* over $V$ if for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.
Hechler forcing
is a set $H_0 = \{ \langle s, f \rangle : s \in <\omega \omega, f : \omega \to \omega, s \subset f \}$,
with a partial ordering
$\langle s, f \rangle \leq \langle t, g \rangle$ if and only if $t \subseteq s$ & $(\forall n \in \omega) f(n) \geq g(n)$.

adds dominating real
Hence it also adds eventually different, unbounded, and independent real.
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