Classification of new reals

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New Reals

Phrase 'Forcing notion *adds a new real*' means that for any generic filter G over V, the generic extension $V[G]$ contains a new subset $\sigma \subset \omega$

Hence $V[G]$ contains a function $\rho : \omega \to \omega$ which does not belong to groundmodel V.

It is quite common in set theory that under the term 'real' we mean subset of ω. Hence elements of Cantor space $\mathcal{C} = {}^{\omega}\{0,1\}$ are reals as well as the function from ω to ω, i.e. elements from Baire space $\mathcal{N}$ are called reals.

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Let M denote an extension of V.

- $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^\omega \cap V$ both $X \cap Y$ and $Y - X$ are infinite.
- A function $f \in M$, $f \in \omega^\omega$, is a *dominating real* over V if for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.
- A function $h \in \omega^\omega$ in the extension is said to be an *unbounded real* over V if for all $f \in \omega^\omega \cap V$ the set $\{n \in \omega : h(n) > f(n)\}$ is infinite.
- A function $h \in \omega^\omega$ in the extension is said to be an *eventually different* real over V if for all $f \in \omega^\omega \cap V$ the set $\{n \in \omega : h(n) = f(n)\}$ is finite.
- M is an ω^ω*-bounding extension* of V if every $f \in M$, $f \in \omega^\omega$ is dominated by a $g \in \omega^\omega \cap V$.

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Remark.

Each dominating real is eventually different.

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Cohen forcing. *Cohen forcing* is countable atomless ordering and is equivalent to any of the following set

- $Seq = \bigcup\{{}^{n}\omega : n < \omega\}$,
- $Seq_2 = \bigcup\{{}^{n}2 : n < \omega\}$,
- $Fn(\omega, 2) = \{f; f : D \to \{0,1\}, D \in [\omega]^{<\omega}\}$,

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- adds a new real,
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- adds unbounded real,
- does not add an eventually different real, hence cannot add dominating reals.

Random Real

- $(\mathrm{Borel}(2^\omega) - \mathit{Null}, \subseteq)$ is **Random forcing**. The ordering is not separative, its separative quotient is
- $(\mathrm{Borel}(2^\omega)/\mathit{Null}, \subseteq)$. This is *ccc* complete atomless Boolean algebra that carries strictly positive σ-additive measure, $m[U] = m(U)$, for each $U \in \mathrm{Borel}(2^\omega)$.

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Theorem

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Theorem

(i) In Random extension are groundmodel reals meager.
(ii) In Cohen extension are groundmodel reals negligible.

Dominating Real

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Hechler Real

Hechler forcing

is a set $H_0 = \{\langle s, f\rangle : s \in {}^{<\omega}\omega,\ f : \omega \to \omega,\ s \subset f\}$,
with a partial ordering
$\langle s, f\rangle \leq \langle t, g\rangle$ if and only if $t \subseteq s\ \&\ (\forall n \in \omega)\ f(n) \geq g(n)$.

adds dominating real
Hence it also adds eventually different, unbounded, and independent real.

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