

2. ORDERINGS AND BOOLEAN ALGEBRAS

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In this section we shall introduce many notions. Most of them are elementary and probably already known to the reader. We will familiarise ourselves with concepts related to orderings, e.g. dense and predense sets, filters, separative quotients etc. We shall also mention the important relationship between an ordered set and a (complete) Boolean algebra. Then we state the Rasiowa-Sikorski theorem and show an application in the special case of σ -complete algebras. The final paragraph will be devoted to the concept of a Boolean matrix and its use. The whole section will also be sprinkled with important examples of orderings.

2.1 DEFINITION. Ordering. An *ordering* (or an *ordered set*) (P, \leq) consist of a nonempty set P together with a reflexive and transitive binary relation \leq , i.e. $x \leq x$ and $x \leq y$ & $y \leq z \rightarrow x \leq z$. If the binary relation is moreover antisymmetric, i.e. if $x \leq y$ & $y \leq x \rightarrow x = y$, we say that (P, \leq) is a partial order.

If $x \leq y$ we say that x is smaller than y or that x is below y . If x, y are such that for no z both $z \leq x$ and $z \leq y$, we say that x, y are *disjoint* or *incompatible* and denote this by $x \perp y$, otherwise they are *compatible*, denoted by $x \parallel y$. An element x with no disjoint elements below x is called an *atom*.

2.2 DEFINITION. Suppose P is an ordering and $X \subseteq P$.

- (i) **Antichain.** X is an *antichain* or is called *disjoint* if any two distinct elements of X are incompatible. X is a *maximal antichain* if it is an antichain and each element of P is compatible with some element of X .
- (ii) **Dense.** X is called *predense* if any element of P is compatible with some element of X . X is dense if for any $x \in P$ there is an element of X which is below x . X is *predense* (dense) below $x \in P$ if for every element y of P below x there is an element $h \in X$ which is compatible with (below) y .
- (iii) **Open dense.** X is called *open dense* if it is downward closed and dense, i.e. $(\forall p \in P)(\exists q \in X)(q \leq p)$ and $(\forall h \in X)((\leftarrow, h] \subseteq X$, where $(\leftarrow, h] = \{p \in P : p \leq h\}$
- (iv) **Centred.** X is *centred* if for any finite $F \subseteq X$ there is a $z \in X$ which is below each $x \in F$.
- (v) **Filter.** X is a *filter* if it is centred and upward closed, i.e. $(\forall h \in X)([h, \rightarrow) \subseteq X$

COMMENTS AND FACTS

- (a) Note that some authors prefer to use the term *preorder* or *quasiorder* where we use the term ordering or order.
- (b) Linear (total) orders (i.e. any two elements are comparable) and orders having the least element cannot contain disjoint elements.
- (c) A subset X of an ordering (P, \leq) is implicitly ordered by the restriction of \leq to X .

2.3 FACT. Let A be maximal antichain and D a dense set. Then

- (i) A is a predense set and $\bigcup\{(\leftarrow, a] : a \in A\}$ is an open dense set.

(ii) A has a refinement $B \subseteq D$, that is, B is a maximal antichain and for all $b \in B$ there is an $a \in A$ with $b \leq a$.

Proof. (hint) For $a \in A$ find a maximal antichain B_a in the ordering $((\leftarrow, a] \cap D, \leq)$. Then it suffices to take $\bigcup\{B_a : a \in A\}$. \square

The following is a first version of the Rasiowa-Sikorski theorem:

2.4 THEOREM. (RASIOWA, SIKORSKI). *Suppose $\{D_n : n < \omega\}$ is a countable family of dense subsets of an ordering P and $p \in P$. Then there is a filter F on P such that*

$$p \in F \quad \& \quad (\forall n < \omega)(D_n \cap F \neq \emptyset).$$

Proof. By induction choose a decreasing sequence $p \geq p_0 \geq p_1 \geq \dots \geq p_n \geq \dots$ such that $p_n \in D_n$. This sequence forms a centred system which can easily be extended to the required filter F . \square

2.5 SEPARATIVE ORDERINGS. We will see that, from the forcing point of view, the disjointness relation of an ordering plays a crucial role apart from the ordering itself.

2.6 NOTATION. *Suppose A is a subset of P . Then we shall write A^\perp to be the set of all elements disjoint with A , formally $A^\perp = \{p \in P : (\forall a \in A)(p \perp a)\}$. In particular $\{x\}^\perp = \{p \in P : x \perp p\}$.*

2.7 DEFINITION. An ordering (P, \leq) is called *separative* if for all $x, y \in P$

$$(*) \quad x \not\leq y \rightarrow (\exists z \leq x)(z \perp y).$$

It is easy to see that $(*)$ can be substituted by any of the following equivalent conditions:

$$(i) \quad x \leq y \equiv \{x\}^\perp \supseteq \{y\}^\perp$$

$$(ii) \quad x \leq y \equiv (\forall z)(z \parallel x \rightarrow z \parallel y)$$

$$(iii) \quad x \leq y \equiv \{y\} \text{ is predense below } x$$

Note that the implications from the left to the right hold in every ordering.

The following definition and fact shows that each ordering can be strengthened to a separative ordering while preserving the disjointness relation:

2.8 DEFINITION. Separative modification. For any ordering \leq on a set P we define its *separative modification* \leq_{sp} as follows:

$$x \leq_{sp} y \equiv \{x\}^\perp \supseteq \{y\}^\perp$$

2.9 FACT. *Let (P, \leq) be an ordering. Then*

$$(i) \quad \leq \subseteq \leq_{sp},$$

$$(ii) \quad \leq_{sp} \text{ is a separative ordering of } P,$$

$$(iii) \quad \text{the disjointness relations for } \leq \text{ and } \leq_{sp} \text{ coincide.}$$

Moreover conditions (i-iii) determine the relation \leq_{sp} uniquely.

From this point of view linear orderings or orderings with a smallest element are not very interesting because in their separative modification the disjointness relation is trivial (i.e. each two elements are comparable) and their quotient (see 2.11) is just the one element partial order.

Every ordering gives rise to a natural equivalence relation and quotient partial order:

2.10 FACT. If (P, \leq) is an ordering we let $x \approx y \equiv x \leq y \ \& \ y \leq x$. Then

- (i) \approx is an equivalence relation and the quotient $(P, \leq)/\approx$ is a partial order.
- (ii) If (P, \leq) is a separative ordering, then the quotient $(P, \leq)/\approx$ is also separative.
- (iii) A dense subset of a separative ordering is again a separative ordered set.

2.11 SEPARATIVE QUOTIENT. The equivalence relation from the preceding fact defined from \leq_{sp} is denoted by \approx_{sp} . The quotient partial order $(P, \leq_{sp})/\approx_{sp}$ is called the *separative quotient* of (P, \leq) .

The following is trivial

2.12 FACT. (P, \leq) is a separative partial order iff \approx_{sp} is the identity on P .

2.13 FACT. If (P, \leq) is an ordering then

$$x \approx_{sp} y \equiv (\forall H \subseteq P)(H \text{ is predense below } x \text{ iff } H \text{ is predense below } y).$$

2.14 EXAMPLES OF ORDERINGS. The properties of orderings relevant for forcing only depend on the combinatorial properties of dense sets. The following examples illustrate that there are nonisomorphic orderings with isomorphic dense subsets (and hence, as will be seen, have isomorphic generic extensions).

First we mention two cardinal characteristics of an ordering:

2.15 DEFINITION. An ordering P is called

- (i) κ -**cc** (or satisfies the κ -**chain condition**) if any antichain of P has cardinality $< \kappa$. If $\kappa = \omega_1$ we say that P is *ccc*.
- (ii) κ -**closed** if for any \leq -descending sequence of elements of P which is of length $< \kappa$ has a lower bound in P . If $\kappa = \omega_1$ we say that P is *σ -closed*.

2.16 COUNTABLE ORDERINGS. Consider the following three sets of finite functions:

- (a) $\text{Seq} = \bigcup \{ {}^n \omega : n < \omega \}$,
- (b) $\text{Seq}_2 = \bigcup \{ {}^n 2 : n < \omega \}$,
- (c) $\text{Fn}(\omega, 2) = \{ f; f : D \rightarrow \{0, 1\}, D \in [\omega]^{<\omega} \}$,

all ordered by inverse inclusion (i.e. $f \leq g$ iff $f \supseteq g$). These systems are countable separative partial orders without atoms (atomless) with a largest element which is the empty set.

2.17 PROPOSITION. Any two countable atomless orderings have isomorphic dense subsets.

Proof. We show that each such order has a countable dense subset T such that (T, \geq) is isomorphic to the full ω -ary tree $\bigcup \{ {}^n \omega : n \in \omega \}$ ordered by inclusion. This clearly suffices. First we enumerate $P = \{ p_n : n < \omega \}$ and for each $p \in P$ we choose an infinite maximal antichain A_p below p . To build T proceed by induction on ω building at step $n + 1$ the $(n + 1)$ -st level T_{n+1} of T as follows: $T_{n+1} = \bigcup \{ A_{p'} : p \in T_n \}$ where $p' \leq p_n, p$ if p_n and p are compatible, otherwise $p' = p$. \square

2.18 A natural generalisation of (c) are the two following separative partial orders (X is an infinite set)

$$(d) \text{Fn}(X, \omega) = \{f; f : D \rightarrow \omega, D \in [X]^{<\omega}\},$$

$$(e) \text{Fn}(X, \{0, 1\}) = \{f; f : D \rightarrow \{0, 1\}, D \in [X]^{<\omega}\},$$

which again have isomorphic dense subsets:

Proof. When $|X| = \omega$ see above. Otherwise partition X into countable sets A_ξ , $\xi < \kappa$. Using Proposition 2.17 fix H_ξ, K_ξ dense subsets of $F(A_\xi, \omega)$ and $F(A_\xi, 2)$ respectively and an isomorphism ϕ_ξ of H_ξ onto K_ξ . Now the set $\{f \in F(X, \omega) : (\forall \xi < \kappa)(f \upharpoonright A_\xi \in H_\xi)\}$ is dense in $F(X, \omega)$ and is isomorphic to the corresponding dense set in $F(X, 2)$ via the (concatenated) isomorphisms ϕ_ξ . \square

2.19 DEFINITION. σ -centred. An ordering P is σ -centred if it can be written as a countable union of filters.

Each σ -centred ordering is ccc but not vice versa, see below. Also note that each countable ordering is σ -centred and the following proposition limits the cardinality from above:

2.20 PROPOSITION. Any σ -centred separative partial order P is of cardinality at most 2^{\aleph_0} .

Proof. Let $P = \bigcup P_n$ where P_n is a maximal centred family. Then the mapping $\varphi : P \rightarrow \mathcal{P}(\omega)$ defined by $\varphi(p) = \{n : p \in P_n\}$ is one-to-one. \square

2.21 PROPOSITION. Both of the orders defined in (d,e) are ccc.

Proof. We shall only show the ccc property for $F(X, 2)$. This is enough, since $F(X, 2)$ and $F(X, \omega)$ have isomorphic dense subsets. In fact we shall show that $F(X, 2)$ satisfies an even stronger condition, namely we will partition $F(X, 2)$ into countably many sets P_n each of which will contain antichains of size at most 2^n . This clearly suffices. Put $P_n = \{f \in F(X, 2) : |\text{dom}f| = n\}$. Take $f_0, \dots, f_{2^n} \in P_n$. We show that at least two are compatible. Put $K = \bigcup \{\text{dom}f_i : i \leq 2^n\}$. Consider the normalised counting measure m on $\mathcal{P}(^K 2)$ and let $X_i = \{g; g : K \rightarrow 2, g \supseteq f_i, |K| < \omega\}$. Then $X_i \cap X_j = \emptyset$ whenever $f_i \perp f_j$. Also $m(X_i) = 1/2^n$ and, using the additivity of measure, we see that at least two of these sets are not disjoint: otherwise $\sum_{i=0}^{2^n} m(X_i) > 1$. \square

Moreover both of the orders (d,e) are σ -centred iff $|X| \leq 2^{\aleph_0}$.

2.22 COUNTABLE FUNCTIONS. Consider the following two families of (at most) countable functions:

$$(f) \text{Fn}(\omega_1, \{0, 1\}, \omega_1) = \{f; f : D \rightarrow \{0, 1\} : D \in [\omega_1]^{<\omega_1}\},$$

$$(g) \text{Fn}(\omega_1, \mathbb{R}, \omega_1) = \{f; f : D \rightarrow \mathbb{R} : D \in [\omega_1]^{<\omega_1}\},$$

ordered by reverse inclusion. \mathbb{R} is the real line. Both of these orders are separative partial orders which are σ -closed.

2.23 PROPOSITION. The orderings (f) and (g) have isomorphic dense subsets and satisfy $(2^{\aleph_0})^+ \text{-cc}$. Also below each of their elements there is an antichain of size 2^{\aleph_0} .

Proof. (hint) We show that (g) can be embedded onto a dense subset of (f): choose a partition $\{A_\xi : \xi < \omega_1\}$ of ω_1 into countable sets and fix a 1-to-1 onto mapping $\phi_\xi : \mathbb{R} \rightarrow 2^{A_\xi}$. Then define $\phi : \text{Fn}(\omega_1, \mathbb{R}, \omega_1) \rightarrow \text{Fn}(\omega_1, 2, \omega_1)$ as follows: $\phi(f) = \bigcup \{\phi_\xi(f(\xi)) : \xi \in \text{dom}f\}$. It is clear that the range of ϕ is a dense subset of $\text{Fn}(\omega_1, 2, \omega_1)$. \square

2.24 COLLAPSING ORDERING. Recall that $\text{Seq}(\alpha)$ is the set of finite sequences of ordinals $< \alpha$. The *collapsing ordering* is $(\text{Seq}(\kappa), \supseteq)$. It is a separative partial order, mainly interesting only in the case that $\omega < \alpha$. For $\alpha = 2, \omega$ it coincides with the orderings defined in 2.16. If $\omega \leq \alpha$ then $\text{Seq}(\alpha)$ has cardinality $|\alpha|$ and is $|\alpha|^+$ -cc.

We shall see (2.55) that for infinite κ the collapsing ordering $\text{Seq}(\kappa)$ is ‘characterised’ by the following properties

- (i) It has cardinality κ ,
- (ii) it is a countable union of maximal antichains, namely $\{^n \kappa : n < \omega\}$,
- (iii) each element is compatible with κ -many members of one of the maximal antichains from (ii).

2.25 LEVI COLLAPSE. The underlying set for the Levi collapse $\text{Levi}(\omega, < \kappa)$ consists of finite functions p from $\kappa \times \omega$ into the ordinals such that $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \text{dom } p$. The ordering is by reverse inclusion, i.e. $p \leq q$ iff $p \supseteq q$.

2.26 DEFINITION. Product of orderings. Suppose $\langle P_i : i \in I \rangle$ are orderings. We define the product ordering.

$$\prod_{i \in I} P_i = \{f : \text{dom } f \in [I]^{< \omega}, (\forall i \in \text{dom } f)(f(i) \in P_i)\},$$

with the ordering defined by

$$f \leq g \equiv \text{dom } f \supseteq \text{dom } g \ \& \ (\forall i \in \text{dom } g)(f(i) \leq_i g(i))$$

Note that in the literature, it is often assumed that each P_i has a maximal element $\mathbf{1}$ and the product is then defined to be the subset of the normal Cartesian product where each function has only finitely many values different from $\mathbf{1}$, i.e. has finite support.

2.27 FACT. $\prod_{\alpha < \kappa} (\text{Seq}(\alpha), \supseteq)$ is isomorphic to a dense subset of $\text{Levi}(\omega, < \kappa)$.

2.28 PROPOSITION. If κ is uncountable regular then $\text{Levi}(\omega, < \kappa)$ is κ -cc and below each element there is an antichain of size λ for any $\lambda < \kappa$.

Before we prove the proposition we prove the following theorem which is a very useful tool.

2.29 THEOREM. (Δ -SYSTEM LEMMA). Suppose $\kappa > \omega$ is a regular cardinal and $\langle A_\alpha : \alpha < \kappa \rangle$ are finite sets. Then there exists a Δ -system of cardinality κ , i.e. a set $I \in [\kappa]^\kappa$ and a kernel K of the Δ -system such that for each $\alpha, \beta \in I, \alpha \neq \beta$ we have $A_\alpha \cap A_\beta = K$.

In particular, every uncountable system of finite sets contains an uncountable Δ -system.

Proof. Shrinking if necessary, we can assume each A_α has the same cardinality, say $n < \omega$. We continue the proof by induction on n . If $n = 1$, then either there is an x such that for κ -many α 's we have $A_\alpha = \{x\}$ or we can choose κ -many distinct A_α 's. Assume $n > 1$ and the theorem holds for all $k < n$. Write $A_\alpha = B_\alpha \cup \{x_\alpha\}$. By induction apply the theorem to the system $\langle B_\alpha : \alpha < \kappa \rangle$ to find I . Then apply the theorem again, this time to $\langle \{x_\alpha\} : \alpha \in I \rangle$ to obtain I' . Now $\langle B_\alpha \cup \{x_\alpha\} : \alpha \in I' \rangle$ is the required Δ -system. \square

Proof. OF PROPOSITION 2.28. We use the Δ -system lemma. Suppose for a contradiction that $\{p_\alpha : \alpha < \kappa\} \subseteq L = \text{Levi}(\omega, < \kappa)$ is a disjoint system. Apply the Δ -system lemma to the domains of the p_α 's. We obtain an index set $A \in [\kappa]^\kappa$ and the root K of the delta system. Note that if any two $p, q \in L$ disjoint, then they are also disjoint when restricted to the intersection of their domains. It follows, that $\{p_\alpha \upharpoonright K : \alpha \in A\}$ is still a disjoint system, so in particular $p_\alpha \upharpoonright K \neq p_\beta \upharpoonright K$ for any $\alpha, \beta \in A$. But this is a contradiction, since the number of distinct elements in L having domain K is $\leq (\omega \cdot \max K)^{< \omega} < \kappa = |A|$. \square

2.30 ($[\omega]^\omega, \subseteq$), INFINITE SUBSETS OF ω . The important partial ordered set $([\omega]^\omega, \subseteq)$ is an example of an ordering, that is neither separative nor σ -closed, while the separative modification is σ -closed. We shall describe the separative modification. For $A, B \in [\omega]^\omega$ it is easy to see that

$$A \perp B \equiv |A \cap B| < \omega,$$

therefore

$$A \subseteq_{\text{sp}} B \equiv |A \setminus B| < \omega \quad \& \quad A \approx_{\text{sp}} B \equiv |A \Delta B| < \omega.$$

2.31 STAR NOTATION. In this special case, the following *star-notation* is usually employed. We write $A \subseteq^* B$ and $A =^* B$ instead of $A \subseteq_{\text{sp}} B$ and $A \approx_{\text{sp}} B$, respectively.

A similar 'Star' notation is also used in the context of functions in ${}^\omega\omega$, where we write

$$\begin{aligned} f \leq^* g &\equiv \{ \{ n : f(n) > g(n) \} \} < \omega, \\ f <^* g &\equiv \{ \{ n : f(n) \geq g(n) \} \} < \omega. \end{aligned}$$

The ordering $({}^\omega\omega, \leq^*)$ is frequently used and is the object of ongoing research.

2.32 FACT. $([\omega]^\omega, \subseteq^*)$ is a σ -closed separative ordering. Its separative quotient, after adding a smallest element, is the quotient Boolean algebra $\mathcal{P}(\omega)/\text{fin}$, where fin is the ideal of finite subsets of ω .

Proof. Suppose $\langle A_n : n < \omega \rangle$ is a \subseteq^* -descending sequence. By induction choose $a_n \in \bigcap_{i < n} A_i \setminus \{a_0, \dots, a_{n-1}\}$. Then $\{a_n : n < \omega\}$ is the required lower bound for $\langle A_n : n < \omega \rangle$. \square

2.33 DEFINITION. AD and MAD families. Antichains and maximal antichains in $([\omega]^\omega, \subseteq)$ are usually called *almost disjoint (AD)* and *maximal almost disjoint (MAD)* families on ω respectively.

2.34 FACT. (i) No MAD family on ω is countable.

(ii) There is an AD family on ω of size 2^{\aleph_0} .

Proof. (i) Let $\{A_n : n < \omega\}$ be an AD family. Choose $a_n \in A_n \setminus \bigcup_{i < n} A_i$. Then if $A = \{a_n : n < \omega\}$ we find that $\{A_n : n < \omega\} \cup \{A\}$ is still an AD family. (ii) We shall construct the AD family on $\text{Seq}(2)$ which clearly suffices, since $|\text{Seq}(2)| = \omega$. For each such branch $f \in 2^\omega$, we let $A_f = \{f \upharpoonright n : n < \omega\} \subseteq \text{Seq}(2)$ be the branch of $\text{Seq}(2)$ corresponding to f . Then $\{A_f : f \in 2^\omega\}$ is the required AD family. \square

2.35 DEFINITION. Boolean algebra. A Boolean algebra $\mathbb{B} = (B, \vee, \wedge, -, \mathbf{0}, \mathbf{1})$ is a set B together with the Boolean operations $a \vee b$ (join), $a \wedge b$ (meet), $-a$ (complement) and two distinguished elements $\mathbf{0}$ and $\mathbf{1}$ satisfying the following:

(1) (commutativity) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$

$$(2) \text{ (distributivity) } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$(3) \text{ (neutrality) } x \vee \mathbf{0} = x, \quad x \wedge \mathbf{1} = x.$$

$$(4) \text{ (complement) } x \wedge \neg x = \mathbf{0}, \quad x \vee \neg x = \mathbf{1}.$$

$$(5) \text{ (nontriviality) } \mathbf{0} \neq \mathbf{1}.$$

Boolean algebras are distributive, complementary lattices. Note that the axioms of Boolean algebra as presented are dual. Since different authors use different set of axioms, we sketch a proof for some of common formulas that are usually considered as axioms.

2.36 LEMMA. *For each x, y, z elements of Boolean algebra the following hold*

$$(i) \quad x \wedge \mathbf{0} = \mathbf{0} \quad x \vee \mathbf{1} = \mathbf{1},$$

$$(ii) \text{ **absorption** } x \vee (x \wedge y) = x \quad x \wedge (x \vee y) = x,$$

$$(iii) \text{ **associativity** } x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

Proof. Equalities in the same row are dual, so it suffices to prove one of them.

$$(i) \quad x \wedge \mathbf{0} = (x \wedge \mathbf{0}) \vee \mathbf{0} = (x \wedge \mathbf{0}) \vee (x \wedge \neg x) = x \wedge (\mathbf{0} \vee \neg x) = x \wedge \neg x = \mathbf{0}.$$

$$(ii) \quad x = x \wedge \mathbf{1} = x \wedge (\mathbf{1} \vee y) = (x \wedge \mathbf{1}) \vee (x \wedge y) = x \vee (x \wedge y).$$

(iii) To show the associativity we start with the conjunction of both formulas

$$(x \wedge (y \wedge z)) \vee ((x \wedge y) \wedge z) = [x \vee ((x \wedge y) \wedge z)] \wedge [(y \wedge z) \vee ((x \wedge y) \wedge z)] = x \wedge [y \wedge z],$$

$$\text{since } [x \vee ((x \wedge y) \wedge z)] = (x \vee (x \wedge y)) \wedge (x \vee z) = x \wedge (x \vee z) = x,$$

$$\text{and } [(y \wedge z) \vee ((x \wedge y) \wedge z)] = [y \vee ((x \wedge y))] \wedge z = y \wedge z.$$

Now we use commutativity to obtain that the same conjunction is equal to $(x \wedge y) \wedge z$

$$((x \wedge y) \wedge z) \vee (x \wedge (y \wedge z)) = [(x \wedge y) \vee (x \wedge (y \wedge z))] \wedge [z \vee ((x \wedge y) \wedge z)] = [x \wedge y] \wedge z.$$

□

The canonical ordering on a Boolean algebra is: $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$ and hence $\mathbf{0}$ is the smallest element and $\mathbf{1}$ the largest in this ordering. We let $\mathbb{B}^+ = \mathbb{B} \setminus \{\mathbf{0}\}$. Then (\mathbb{B}^+, \leq) is a separative partial ordering.

We shall frequently use notions and properties defined for partial orders also in the context of Boolean algebras. In such cases we always refer to the partial order (\mathbb{B}^+, \leq) . This order is separative, since if $a \not\leq b$ then we can let $c = a - b \neq \mathbf{0}$ and $c \leq a$ while $c \wedge b = \mathbf{0}$, where $a - b = a \wedge (\neg b)$.

2.37 DEFINITION. Density. The *density* of a Boolean algebra $\pi(\mathbb{B})$ (also called π -weight) is the minimal cardinality of a dense subset of (\mathbb{B}^+, \leq) .

2.38 DEFINITION. Complete Boolean algebra.

(i) A Boolean algebra is *complete* if every subset $X \subseteq \mathbb{B}$ has an infimum in the canonical ordering, denoted by $\bigwedge X$. In this case every subset also has a supremum, denoted by $\bigvee X$ and the standard de Morgan laws hold. We say that “ \mathbb{B} is a cBA” as a shorthand for “ \mathbb{B} is a complete Boolean algebra”.

(ii) A Boolean algebra is σ -*complete* if every countable subset has a meet (or join).

2.39 DEFINITION. Generators. We say that a subset $X \subseteq B^+$ of a complete Boolean algebra B *completely generates* B if there is no proper complete subalgebra of B containing X . We define

$$g(B) = \min\{|X| : X \subseteq B^+, X \text{ completely generates } B^+\}.$$

Similarly for a σ -complete algebra we define a σ -*completely generating set* and correspondingly $g_\omega(B)$, the smallest cardinality of a set which σ -completely generates B .

The power-set $\mathcal{P}(X)$ of a nonempty set X is a typical example of a complete atomic Boolean algebra. The singletons are atoms and the operation of meet and join correspond to intersection and union respectively with $\mathbf{0} = \emptyset$, $\mathbf{1} = X$. Subalgebras of this algebra are called *fields* of sets. If they are moreover σ -subalgebras, then they are called σ -fields of sets.

2.40 EXAMPLE. Given a topological space (S, τ) there are several natural algebras determined by (S, τ) :

- (i) The algebra $\text{Clop}(S)$ of closed-and-open, *clopen* for short, subsets of S .
- (ii) The algebra $\text{Baire}(S)$ is the smallest σ -field which contains all zero sets from S . A subset of S is a *zero set* iff it is the preimage of $\{0\}$ under some continuous mapping $f : S \rightarrow \mathbb{R}$.
- (iii) The algebra $\text{Borel}(S)$ is the smallest σ -field which contains all open subsets of S .
- (iv) The algebra $\text{BP}(S)$ consisting of all sets with the *Baire property*. This is the smallest σ -field containing all meagre sets and all open sets.

It is evident that $\text{Clop}(S) \subseteq \text{Baire}(S) \subseteq \text{Borel}(S) \subseteq \text{BP}(S)$.

2.41 STONE THEOREM. Given a Boolean algebra B there is a compact, zero-dimensional, Hausdorff space S such that $\text{Clop}(S)$ is isomorphic to B . It follows that B is isomorphic to a field of sets.

Note, however, that not every σ -complete Boolean algebra is isomorphic to a σ -field of sets. The following holds:

2.42 LOOMIS, SIKORSKI THEOREM. Every σ -complete Boolean algebra is isomorphic to the quotient F/\mathcal{I} , where F is a σ -field of sets and \mathcal{I} is a σ -ideal on F .

The previous theorem is an easy consequence of the fact that the σ -complete algebra $\text{Baire}(2^\kappa)$ is a free object on κ -many free generators in the category of σ -complete Boolean algebras. In the category of all Boolean algebras, the free objects on κ -many generators are the algebras $\text{Clop}(2^\kappa)$. Also note that for $\kappa > \omega$, $\text{Baire}(2^\kappa) \subsetneq \text{Borel}(2^\kappa)$.

2.43 FROM ORDERED SETS TO COMPLETE BOOLEAN ALGEBRAS. The aim of the following sections is to explain the following theorem which, it is hoped, will help to better understand what is essential for forcing.

2.44 THEOREM. *Suppose (P, \leq) is an ordering. Then there exists a unique (up to isomorphism) complete Boolean algebra B and a function $h : P \rightarrow B^+$ such that the following holds:*

- (i) h preserves the ordering,
- (ii) h preserves the disjointness relation and

(iii) the image of P is dense in B^+ .

The (unique) algebra B is denoted $RO(P)$. Note that h need not be an embedding.

Observe that any dense subordering of P determines the same algebra:

Suppose $H \subseteq P$ and $h : P \rightarrow B$ satisfies (i),(ii),(iii). Then the restriction of h to H also satisfies (i), (ii), (iii).

This is the reason why we have been concerned with isomorphisms between dense subsets of (formally) nonisomorphic orderings.

If h satisfies (iii) and $H = h[P]$, then $(H, \leq_{\mathbb{B}})$ with the order inherited from the Boolean algebra is a separative partial order. If we define $x \preceq y \equiv h(x) \leq_{\mathbb{B}} h(y)$ then $\preceq = \leq_{sp}$, the separative modification of (P, \leq) and, moreover, $(P, \leq) \approx_{sp}$ is isomorphic to $(H, \leq_{\mathbb{B}})$.

The following simple lemma illustrates the uniqueness of $RO(P)$.

2.45 LEMMA. *If two complete Boolean algebras B and C have isomorphic dense subsets, then they are isomorphic. Moreover any isomorphisms of dense subsets can be extended into isomorphisms of the whole algebras.*

We shall now show two approaches to finding the unique algebra $RO(P)$. The first approach is algebraical, the second topological. We shall also show that they have a common core.

ALGEBRAICAL APPROACH

In the first approach $RO(P)$ will consist of downward closed subsets of P and the canonical ordering will coincide with inclusion. The complement operation will not coincide with set complement and, moreover, $RO(P)$ will generally not be a σ -field of sets.

2.46 DEFINITION (Regularisation). Recall that for any set $A \subseteq P$ we have defined A^\perp as $\{x \in P : (\forall y \in A)(x \perp y)\}$. We define the *regularisation* of A as follows:

$$\text{reg}(A) = (A^\perp)^\perp$$

A set will be called *regular* if $\text{reg}(A) = A$.

Note that $A \subseteq (A^\perp)^\perp$ and also

$$\text{reg}(A) = \{p \in P : A \text{ is predense below } p\}.$$

From this it follows that $((A^\perp)^\perp)^\perp = A^\perp$ and also that the regularisation $\text{reg}(A)$ is always a regular set. Moreover A^\perp is regular for any $A \subseteq P$.

We are now ready to define $RO(P)$:

$$RO(P) = \{A \subseteq P : \text{reg}(A) = A\}.$$

Further define $h : P \rightarrow RO(P)$ as $h(p) = \text{reg}(\{p\})$. Since both \emptyset and P are regular subsets of P we can let $\mathbf{0} = \emptyset$ and $\mathbf{1} = P$. Also the intersection of regular sets is regular so the Boolean operation \wedge coincides with set intersection. We let $A \vee B = \text{reg}(A \cup B)$ and $\neg A = A^\perp$. It is clear that $RO(P)$ with the described operations is Boolean algebra. The following holds for any system $\{A_i : i \in I\} \subseteq RO(P)$:

$$\bigwedge \{A_i : i \in I\} = \text{reg}\left(\bigcap_{i \in I} A_i\right)$$

and hence $RO(P)$ is a complete Boolean algebra. The mapping h is a homomorphism of P onto a dense subset of $RO(P)$ and it can also be described as

$$h(p) = \text{reg}(\{\leftarrow, p\}),$$

where $\{\leftarrow, p\} = \{q \in P : q \leq p\}$.

TOPOLOGICAL APPROACH

Let (S, τ) be a topological space. We always assume S to be nonempty. $\text{Open}^+(S)$ shall denote the collection of all nonempty open sets. We know that the ordering $P = (\text{Open}^+(S), \subseteq)$ determines a complete Boolean algebra $\text{RO}(P)$. We shall now try to describe it in a topological language.

2.47 DEFINITION. An subset G of S is *regular open* if it is equal to the interior of its closure, i.e. iff $G = \text{int cl } G$. The collection of regular open subsets of a space is customarily denoted by $\text{RO}(S)$. It is evident, that $\text{RO}^+(S) \subseteq \text{Open}^+(S)$ and $\emptyset, X \in \text{RO}(S)$.

For any set $H \subseteq S$ we define its *regularisation* $r(H) := \text{int cl } H$. It is clear that $r(H)$ is a regular open set.

To motivate the previous definition of $r(H)$ somewhat, consider the following operation:

$$\{H\}^\perp = \{U \in \text{Open}^+ : U \cap H = \emptyset\}$$

Then $V = \bigcup H^\perp$ is open, $\text{cl } H = S \setminus V$ and $r(H) = \text{int } S \setminus V$. The analogy with the algebraical construction should be clear.

2.48 THEOREM. *The set $\text{RO}(S)$ endowed with the operations $A \vee B = r(A \cup B)$, $A \wedge B = r(A \cap B)$ and $-A = r(S \setminus A)$ forms a Boolean algebra. The canonical ordering of this Boolean algebra coincides with set inclusion and the following holds:*

(i) $-A = \text{int } (S \setminus A)$

(ii) $r(A \cap B) = A \cap B$ for all $A, B \in \text{RO}(S)$.

(iii) $\text{RO}(S)$ is a complete Boolean algebra and for $\mathcal{A} \subseteq \text{RO}(S)$ the infinite meet $\bigwedge \mathcal{A}$ is exactly equal to $\text{int } \bigcap \mathcal{A} = r(\bigcap \mathcal{A})$.

Proof. (i) is clear. For (ii) it suffices to show that $r(A \cap B) \subseteq A \cap B$, the other direction is clear. Since $r(A \cap B)$ is an open set contained in $\text{cl } A \cap B \subseteq \text{cl } A$ we have that $r(A \cap B) \subseteq r(A) = A$. Similarly $r(A \cap B) \subseteq B$. We now prove (iii). Given $\mathcal{A} \subseteq \text{RO}(S)$ we show that the regularisation $D = r(\bigcap \mathcal{A})$ is the Boolean meet since. It is clearly a lower bound for \mathcal{A} . If C is a different lower bound, then it is regular open and $C \subseteq A$ for each $A \in \mathcal{A}$, then $C \subseteq \bigcap \mathcal{A}$ and since r is monotone with respect to inclusion, also $C \subseteq D$. We leave the rest of (iii) as an exercise for the interested reader. \square

2.49 CONCLUSION. If \mathcal{B} is a basis for the topology of S or \mathcal{P} is a π -basis for the topology of S then both \mathcal{B} and \mathcal{P} are dense in $(\text{Open}^+, \subseteq)$ and

$$\text{RO}(\text{Open}^+, \subseteq) \simeq \text{RO}(\mathcal{B}, \subseteq) \simeq \text{RO}(\mathcal{P}, \subseteq) \simeq \text{RO}(S).$$

2.50 RELATIONSHIP BETWEEN THE APPROACHES. Given an order (P, \leq) we can define a topology on P by giving the base

$$\{(\leftarrow, p] : p \in P\}$$

Regular open sets in this topology are exactly the regular sets arising in the algebraical approach.

2.51 COMPLETION. For any Boolean algebra B , there is a complete Boolean algebra \bar{B} such that B^+ is a dense subset of \bar{B} . It suffices to take $RO(B^+)$. The complete algebra \bar{B} is called regular completion or shortly completion.

2.52 DEFINITION. A topological space has the *Baire category property* if no nonempty open set is meager.

Equivalently a space has the Baire category property iff the intersection of countably many open dense subsets is a dense set. The following theorem is a classical theorem of general topology

2.53 THEOREM. *If a space S has the Baire category property, then*

$$RO(S) \simeq \text{Borel}(S)/\text{Meager} \simeq \text{BP}(S)/\text{Meager}$$

For proof see [BS89] or [Eng89].

2.54 A CHARACTERIZATION OF COLLAPSING ALGEBRAS The collapsing algebra for an infinite cardinal κ , denoted $\text{Col}(\omega, \kappa)$, is a complete Boolean algebra determined by the separative partial order $(\text{Fn}(\omega, \kappa), \supseteq)$ defined in 2.16. Since the set ${}^{<\omega}\kappa$ of all finite sequences of ordinals $< \kappa$ is dense in $\text{Col}(\omega, \kappa)$ it is easy to see that $\text{Col}(\omega, \kappa)$ is isomorphic to the algebra of all regular open subsets of the completely metrisable space κ^ω , the topological product of ω many copies of κ with the discrete topology.

2.55 THEOREM (McAloon). *Let P be a separative partial order. The complete Boolean algebra $B = RO(P)$ is isomorphic to $\text{Col}(\omega, \kappa)$ if and only if*

- (i) $|P| = \kappa$ and
- (ii) *there are countably many maximal antichains $\{\mathcal{A}_n : n < \omega\}$ such that for every $p \in P$ there is $n < \omega$ so that p is compatible with κ -many elements of \mathcal{A}_n .*

The proof of the theorem uses the following “trick”, which is a useful fact on its own:

2.56 LEMMA. *Suppose (P, \leq) is a partial order, $\mathcal{A} \subseteq P$ an antichain and $\langle p_\alpha : \alpha < \kappa \rangle$ a system of elements of P . Moreover assume that each p_α is compatible with κ -many elements of \mathcal{A} . Then there is an antichain \mathcal{B} refining \mathcal{A} such that for each α*

$$\kappa \leq |\{x \in \mathcal{B} : x \leq p_\alpha\}|.$$

Proof. Let $A_\alpha = \{a \in \mathcal{A} : p_\alpha \parallel a\}$. Since each $|A_\alpha| \geq \kappa$ it is easy to find pairwise disjoint sets B_α such that $B_\alpha \in [A_\alpha]^\kappa$. For each $a \in B_\alpha$ we can find b_a which is below both a and p_α . Now let

$$\mathcal{B} = \bigcup_{\alpha < \kappa} (\{b_a : a \in B_\alpha\} \cup A_\alpha \setminus B_\alpha).$$

□

This lemma allows us to construct a subset of P isomorphic to $({}^{<\omega}\kappa \setminus \{0\}, \supseteq)$ and the required isomorphism with $\text{Col}(\omega, \kappa)$ is easily derived.

As an application of the theorem of McAloon, we mention the following important property of $\text{Col}(\omega, \kappa)$ (for proof see [Rv89]).

2.57 THEOREM (Kripke). Universality.

- (i) Given an ordering (P, \leq) of size at most κ there is a homomorphism $h : P \rightarrow (\text{Col}(\omega, \kappa)^+, \leq)$ which preserves disjointness and maximal antichains.
- (ii) Given any complete Boolean algebra B which has a dense subset of cardinality $\leq \kappa$, i.e. $\pi(B) \leq \kappa$, there is an embedding h of B into $\text{Col}(\omega, \kappa)$ such that any automorphism of the image $h[B]$ can be extended to the whole algebra $\text{Col}(\omega, \kappa)$.
- (iii) (R. Solovay) Moreover $\text{Col}(\omega, \kappa)$ is a complete boolean algebra of size 2^κ with countable many complete generators.

2.58 DEFINITION. Matrix. Let B be a Boolean algebra and κ, λ cardinal numbers. We call $\mathbb{A} = (a_{\alpha, \beta} : \alpha < \kappa, \beta < \lambda)$ a (κ, λ) -matrix in B if

- (i) the elements of the rows are pairwise disjoint, i.e. for $\alpha < \kappa$ and $\mu, \nu < \lambda, \mu \neq \nu$ we have $a_{\alpha, \mu} \wedge a_{\alpha, \nu} = \mathbf{0}$, and
- (ii) the joins of the rows is $\mathbf{1}$, i.e. for each $\alpha < \kappa, \bigvee \{a_{\alpha, \beta} : \beta < \lambda\} = \mathbf{1}$.

In other words, a (κ, λ) -matrix is a system of κ disjoint partitions of unity each of size λ (note that we admit zero elements).

- (iii) (κ, λ) -matrix A is called *surjective* if moreover $\bigvee \{a_{\alpha, \beta} : \alpha < \kappa\} = \mathbf{1}$ for each $\beta < \lambda$.
- (iv) (κ, λ) -matrix A is called *bijective* if the transposed family $A = (a_{\beta, \alpha} : \alpha < \kappa, \beta < \lambda)$ is also a matrix.

2.59 EXAMPLE. Consider the collapsing algebra $\text{Col}(\omega, \kappa)$ which is isomorphic with $\text{RO}(\kappa^\omega)$. For $n < \omega, \beta < \kappa$ define

$$a_{n, \beta} = \{f \in {}^\omega \kappa : f(n) = \beta\}.$$

This is an (ω, κ) -surjective matrix in $\text{RO}(\kappa^\omega)$.

We shall see later that a (κ, λ) -matrix in a complete Boolean algebra is a canonical name for a mapping from κ into λ .

2.60 COMPOSITION. Given cardinals κ, λ, μ a (κ, λ) -matrix \mathbb{A} and a (λ, μ) -matrix \mathbb{B} , then

$$\mathbb{A} * \mathbb{B} = \{c_{\alpha, \gamma} : \alpha < \kappa, \gamma < \mu\},$$

where $c_{\alpha, \gamma} = \bigvee \{(a_{\alpha, \beta} \wedge b_{\beta, \gamma} : \beta < \lambda)\}$ is a (κ, μ) -matrix.

The following proposition is a simple consequence of a forcing consideration and Cantor - Bernstein theorem. Its direct algebraic proof may appear lengthy.

2.61 PROPOSITION. Let $\kappa \leq \kappa_1 < \lambda_1 \leq \lambda$ be cardinal numbers. If there is a surjective (κ, λ) -matrix in a complete Boolean algebra B , then there is also a bijective (κ_1, λ_1) -matrix in B .

2.62 WORKING WITH MATRICES. The Baire number $b(S)$ of a topological space S is the minimal cardinality of a family of nowhere dense subsets of S which cover S . This number is well defined for perfect spaces, i.e. spaces without isolated points.

For a perfect Polish space S the Baire number $b(S)$ is equal to $\text{cov}(\mathcal{M})$ (see Cichoń's diagram) and its value is independent of the axioms of Set Theory. In metric spaces, which are hereditarily nonseparable, i.e. every nonempty open subset has uncountable density, the situation is radically different. We shall show that in this case, the Baire number is ω_1 , the smallest it can be. Examples of such spaces are the generalised Baire spaces $\mathcal{N}(\kappa)$ for κ uncountable or nonseparable Banach spaces with the norm metric. The proof of this intrinsically topological theorem, valid in ZFC, was motivated by forcing techniques.

2.63 THEOREM. *Every hereditarily nonseparable metric space without isolated points can be written as a union of an increasing chain of length ω_1 consisting of nowhere dense sets. So, in particular, the Baire number of such a space is ω_1 .*

Proof. The idea of the proof is based on a construction of a surjective (ω, κ) -matrix for some $\kappa > \aleph_0$ in a complete Boolean algebra $\text{RO}(S)$.

- (a) There is a maximal family of disjoint open subsets of S such that each of the sets is homogeneous in density, so it is sufficient to prove the theorem for spaces which are homogeneous in density $\kappa > \omega$, i.e. spaces where all open subspaces have a dense subset of cardinality κ .
- (b) Since S is a metric space, we know that for any nonempty open subset U the weight, π -weight and density of U coincide. Also, each open set U contains a family of size κ consisting of nonempty disjoint open subsets of U .
- (c) Since each open subset of S has cellularity κ , i.e. contains at least κ -many disjoint open subsets, it is easy to find maximal disjoint collections \mathcal{R}_n ($n < \omega$), of open sets such that

$$\begin{aligned} (\forall n < \omega)(\forall U \in \mathcal{R}_n) \text{diam}(U) &< \frac{1}{n+1}, \\ \mathcal{R}_{n+1} \text{ refines } \mathcal{R}_n \text{ and} \\ (\forall U \in \mathcal{R}_n) |\{V \in \mathcal{R}_{n+1} : V \subseteq U\}| &= \kappa. \end{aligned}$$

- (d) Notice that for each nonempty open $G \subseteq S$ there is some $n < \omega$ and $V \in \mathcal{R}_n$ such that $V \subseteq G$. Indeed, choose $k < \omega$ such that for some $x \in G$ the ball $B(x, 1/k) \subseteq G$. Then for $n = 2k$ we can use the maximality of \mathcal{R}_n together with the requirement that all elements of \mathcal{R}_n have diameter at most $1/(n+1)$.

In other words, $\bigcup_{n < \omega} \mathcal{R}_n$ is a π -basis for the space S . Now comes an important step immediately followed by the heart of the proof.

- (e) Using the \mathcal{R}_n 's it is easy to build a (ω, κ) -matrix $\mathbb{A} = \langle A_{n,\alpha} : n < \omega, \alpha < \kappa \rangle$ of nonempty open sets such that rows are pairwise disjoint and for any nonempty open U there is a row $n < \omega$ such that U meets all elements of $\{A_{n,\alpha} : \alpha < \kappa\}$. Such A is a surjective (ω, κ) -matrix.
- (f) Since $\kappa \geq \omega_1$, by Proposition 2.61 there is (ω_1, ω) matrix $\mathbb{C} = \{C_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ such that the rows are countable maximal disjoint families of nonempty open sets and the columns consist of disjoint open sets. Also note that all of the sets can be chosen to be regular open so we can work in the complete algebra $\text{RO}(S)$.
- (g) We now let $W_\alpha = S \setminus \bigcup \{C_{\alpha,n} : n < \omega\}$ for $\alpha < \omega_1$ and define $F_\alpha = \bigcap_{\alpha \leq \beta} W_\beta$. It is clear that $\langle F_\alpha : \alpha < \omega_1 \rangle$ is an increasing family of nowhere dense closed sets. Also $\bigcup_{\alpha < \omega_1} F_\alpha = S$ for otherwise if $x \in X \setminus \bigcup_{\alpha < \omega_1} F_\alpha$ then in for each $\alpha < \omega_1$ there is an $n < \omega$ such that $x \in C_{\alpha,n}$. So there must be an $n < \omega$ such that for ω_1 -many α 's $x \in C_{\alpha,n}$ but this is impossible, since the columns are disjoint.

□

NOTE.

- (i) The family \mathcal{R}_n described in (c) can be constructed so that $(\bigcup \{\mathcal{R}_n : n < \omega\}, \subseteq)$ is isomorphic to $(\text{Fn}(\omega, \kappa), \supseteq)$ (see definition 2.16) so its completion is isomorphic to $\text{Col}(\omega, \kappa)$.

- (ii) The crucial part of the previous proof consisted in finding the (ω, κ) matrix where $\kappa \geq \omega_1$, the rows are maximal disjoint families of open sets and for each nonempty open set there is a row such that the open set intersects every element of this row. In particular, whenever we can find such a matrix in some space S then S can be written as an increasing union of nowhere dense sets of length ω_1 . Note that if $\text{cf}(\kappa) > \omega$, then the space of uniform ultrafilters on κ , which is far from being metric, has such a matrix. For the genesis of the proof see [BV72], [BS95] and [She07].

2.64 WORKING WITH NAMES. Our aim is to prove a special case of the Loomis-Sikorski theorem and in the course of the proof also demonstrate the techniques of names.

We start with a slightly more general version of the Rasiowa-Sikorski theorem (see 2.4):

2.65 THEOREM. *Let A be a Boolean algebra and $\langle S_n : n < \omega \rangle$ an arbitrary family, where each S_n is of the form $\langle \otimes_n, X_n \rangle$ with $X_n \subseteq A$ and $\otimes_n \in \{\wedge, \vee\}$. Then for any $a \in A^+$ there is a filter F on A such that $a \in F$ and for each $n < \omega$ there is an $a_n \in F$ satisfying*

- (i) if $\otimes_n = \vee$ then either $a_n \leq x$, for some $x \in X_n$ or $a_n \in X_n^\perp$,
(ii) if $\otimes_n = \wedge$ then either a_n is below each $x \in X_n$ or there is an $x \in X_n$ disjoint with a_n .

Proof. Follows from Theorem 2.4, when we put

$$D_n = \begin{cases} \{a \in A^+ : (\exists x \in X_n) a \leq x\} \cup X_n^\perp, & \text{provided that } S_n = \langle \vee, X_n \rangle, \\ \{a \in A^+ : (\forall x \in X_n) a \leq x\} \cup \bigcup \{x^\perp : x \in X_n\}, & \text{provided that } S_n = \langle \wedge, X_n \rangle. \end{cases}$$

□

Consider the Baire space $\mathcal{N} = {}^\omega \omega$ with the standard metric see Example 1.27, and denote \mathbb{B} the σ -field of Borel subsets of \mathcal{N} . Let $b_{i,j} = \{f \in \mathcal{N} : f(i) = j\}$ for $i, j < \omega$. Then $\mathbb{B} = \langle b_{i,j} : i, j < \omega \rangle$ is a matrix in \mathbb{B} consisting of clopen sets which σ -completely generate the field \mathbb{B} .

Given any σ -complete Boolean algebra A with countably many σ -complete generators $\{g_n : n < \omega\}$ it is easy to find a matrix $\mathbb{A} = \langle a_{i,j} : i, j < \omega \rangle$ on A such that for each $i, j < \omega$ either $g_i \perp a_{i,j}$ or $g_i \geq a_{i,j}$. For example it suffices to take $a_{i,0} = g_i$, $a_{i,1} = -g_i$ for each $i < \omega$ and $\mathbf{0}$ for the other elements.

2.66 THEOREM. *The mapping $f : \mathbb{B} \rightarrow \mathbb{A}$ defined by*

$$f(b_{i,j}) = a_{i,j}, \quad i, j < \omega.$$

can be uniquely extended to a σ -complete homomorphism $F : \mathbb{B} \rightarrow A$ onto A . In particular the Boolean algebra A is isomorphic to $\text{Borel}(\mathcal{N})/I$ where $I = \text{Ker}(F)$ is a σ -complete ideal of Borel sets.

2.67 COROLLARY. *(see e.g. [Fre84]) Any complete ccc Boolean algebra with countably many complete generators is isomorphic to a quotient of the algebra $\text{Borel}(\mathcal{N})$ modulo some σ -complete ideal on $\text{Borel}(\mathcal{N})$.*

Proof. OF THE THEOREM 2.66: Put $N_0 = \omega \times \omega$, henceforth we refer the elements of N_0 as a names (of rank zero) for elements of generating matrix in \mathbb{B} and in A respectively. The 'meaning' of a name $\langle i, j \rangle$ in algebra \mathbb{B} is $\|\langle i, j \rangle\|_{\mathbb{B}} = b_{i,j}$ and similarly in algebra A it is $\|\langle i, j \rangle\|_A = a_{i,j}$.

We define the hierarchy of names for elements of the Boolean algebras B and A by transfinite recursion. Names will be denoted by lowercase letters with a dot over them,

$$\begin{aligned} N_1 &= \langle \langle \circledast \rangle \wedge \langle \dot{n}_i : i < \omega \rangle, \dot{n}_i \in \omega \rangle \\ N_{\alpha+1} &= \langle \langle \circledast \rangle \wedge \langle \dot{b}_i : i < \omega \rangle, \dot{b}_i \in N_\alpha \rangle \\ N_\alpha &= \bigcup_{\beta < \alpha} N_\beta, \quad \alpha \text{ limit} \\ N &= \bigcup_{\beta \in \omega_1} N_\beta, \end{aligned}$$

where $\circledast \in \{\vee, \wedge\}$.

For any $\dot{b} \in N$ we define, by recursion, the interpretation $\|\dot{b}\|_B$ of \dot{b} in B using the matrix \mathbb{B}

$$\|\langle \wedge, \dot{n}_i : i < \omega \rangle\|_B = \bigwedge_{i < \omega} \|\dot{n}_i\|_B \quad \& \quad \|\langle \vee, \dot{n}_i : i < \omega \rangle\|_B = \bigwedge_{i < \omega} \|\dot{n}_i\|_B.$$

Similarly we define the interpretation in A using the matrix \mathbb{A} . We say that \dot{b} is a name for $b \in B$ if $\|\dot{b}\|_B = b$.

CLAIM 1. Each element of our algebras has a name, i.e.

$$(\forall b, a \in B, A)(\exists \dot{b}, \dot{a})(\|\dot{b}\|_B = b \ \& \ \|\dot{a}\|_A = a).$$

Claim 1. follows from the fact that $\neg b_{i,j} = \bigvee b_{j,k} : j \neq k$, since \mathbb{B} is a matrix. Hence all interpretation are closed under complement.

Consider relation

$$F = \{ \langle b, a \rangle : (\exists \dot{n} \in N) \|\dot{n}\|_B = b \text{ and } \|\dot{n}\|_A = a \}.$$

Clearly F extends mapping f .

CLAIM 2. F is a mapping.

Let $\dot{b}_1, \dot{b}_2 \in N$ $\dot{b}_1 \neq \dot{b}_2$ be such that $\|\dot{b}_1\|_B = \|\dot{b}_2\|_B$ and $a_1 = \|\dot{b}_1\|_A \neq \|\dot{b}_2\|_A = a_2$. So $a = a_1 \Delta a_2 \neq \mathbf{0}$.

Now consider a countable set of names $S \subset N$ such that

- (i) $\dot{b}_1, \dot{b}_2 \in S$,
- (ii) $\{\langle \vee \rangle \wedge \langle \dot{i}, j \rangle : j < \omega\} \in S$ for each $i \in \omega$,
- (iii) S is closed to names of lower rank, i.e. if $\dot{n} = \langle \circledast \rangle \langle \dot{n}_i : i \in \omega \rangle \in S$ then all $\dot{n}_i \in S$.

Now apply version 2.65 of Rasiova - Sikorski Theorem and take ultrafilter G in A such that $a \in G$ and satisfies conditions of Theorem 2.65. Then there is exactly one mapping $g : \omega \rightarrow \omega$ such that for every $i \in \omega$ $a_{i,g(i)} \in G$.

Denote $U = \{X \in B : g \in X\}$, note that $g \in \mathcal{N}$. U is an ultrafilter in the field B and the following holds $g \in \|\dot{n}\|_B$ iff $\|\dot{n}\|_A \in G$, for each $\dot{n} \in S$.

Therefor $g \in \|\dot{b}_1\|_B$ iff $g \in \|\dot{b}_2\|_B$ iff $a_1 \in G$ iff $a_2 \in G$, but $a = a_1 \Delta a_2 \in G$ and so only one of a_1, a_2 belongs to G , a contradiction. \square

Many of important examples of orderings for adding new real can be expressed as an ordering of type $(\text{Borel}(\mathcal{N}) - \mathcal{I})$, for some σ -complete ideal \mathcal{I} . Such orderings are not separable. The separable quotient is a σ -complete Boolean algebra $\text{Borel}(\mathcal{N})/\mathcal{I}$ with at most countably many generators.

Moreover instead of Baire space \mathcal{N} one can consider any uncountable Polish space P since such spaces are Borel isomorphic, i.e. there is one-to-one mapping $f : \mathcal{N} \rightarrow P$, which is onto and Borel measurable, which means that f induces isomorphism of the fields $\text{Borel}(\mathcal{N})$ and $\text{Borel}(P)$. To study forcing properties for this kind of orderings means to study properties of σ -ideals on Polish spaces and properties of orderings $(P_{\mathcal{I}}, \subseteq)$, where $P_{\mathcal{I}} = \text{Borel}(P) - \mathcal{I}$. This reveals a part of the riddle hidden beyond the title of J. Zapletal book 'Forcing Idealized'.

2.68 CHAINS OF CCC ALGEBRAS. The last section of this chapter will be devoted to a demonstration of the techniques of elementary submodels (see 1.43) and McAloon's characterisation of the collapsing algebras $\text{Col}(\omega, \kappa)$.

2.69 DEFINITION. Chain of algebras. A chain of algebras is a sequence of Boolean algebras $\langle B_{\xi} : \xi < \alpha \rangle$ such that for $\xi_1 < \xi_2$

- (i) B_{ξ_1} is a subalgebra of B_{ξ_2} ,
- (ii) any maximal antichain in B_{ξ_1} is also a maximal antichain in B_{ξ_2} and

It is *continuous* if moreover

- (iii) if $\beta < \alpha$ is limit, then $\bigcup\{B_{\xi} : \xi < \beta\}$ is dense in B_{β} .

The conditions (i) and (ii) say that B_{ξ_1} is a *regular* subalgebra of B_{ξ_2} .

2.70 PROPOSITION (R.Solovay). Let $\langle B_{\xi} : \xi < \alpha \rangle$ be a continuous chain of ccc Boolean algebras. Then $\bigcup\{B_{\xi} : \xi < \alpha\}$ is again a ccc Boolean algebra.

Proof. Let $B = \bigcup\{B_{\xi} : \xi < \alpha\}$. It is sufficient to prove the theorem for the case $\alpha = \omega_1$. Let $A \subseteq B$ be a maximal antichain. We shall show, that $|A| \leq \omega$. Choose κ sufficiently big so that $\mathcal{C} = \langle B_{\xi} : \xi < \alpha \rangle, B, A \in H(\kappa)$ and let $M \preceq \langle H(\kappa), \mathcal{C}, B, A \rangle$ be a countable elementary submodel. Define $A^* = A \cap M$. We shall show that in fact $A^* = A$. Let $\xi = M \cap \omega_1$. Since for each $a \in A$ there is $\delta < \omega_1$ such that $a \in B_{\delta}$ by elementarity we have

$$A^* = \bigcup_{\alpha < \xi} A \cap B_{\alpha} \cap M.$$

Since $A \cap B_{\alpha}$ is an antichain in the ccc Boolean algebra B_{α} it must be countable and hence $A \cap B_{\alpha} \subseteq M$ (see 1.44, iii). Thus

$$(*) \quad A^* = \bigcup_{\alpha < \xi} A \cap B_{\alpha}.$$

We will show that A^* is a maximal antichain in B_{ξ} . Suppose not, then there is an $\alpha < \xi$ and $b \in B_{\alpha}$ such that for all $\beta < \xi$ and all $a \in A \cap B_{\beta}$, $b \perp a$ (here we used *). Working in M we may write this as

$$M \models (\exists \alpha < \omega_1)(\exists b \in B_{\alpha})(\forall \beta < \omega_1)(\forall a \in A \cap B_{\beta})(b \perp a),$$

by elementarity this must also hold in $H(\kappa)$ but that would contradict the fact that A is a maximal antichain.

Since A^* is a maximal antichain in B_{ξ} it must also be maximal in B (recall the continuity of the chain) so $A^* = A$. Since A^* is countable we are finished. □

The following fact, which is the core of the previous proof, deserves a mention on its own (see e.g. [Mek84]):

2.71 PROPOSITION. *An ordering (P, \leq) is ccc iff for each countable $M \preceq H((2^{|P|})^+)$ with $P \in M$ and any dense $D \subseteq P, D \in M$, the trace $D \cap M$ is predense in P .*

Proof. If P is not ccc, then, there is an uncountable antichain $A \subseteq P$. By elementarity, there is such an A with $A \in M$. But $A \cap M$ is countable, so $A \setminus (A \cap M)$ is nonempty so $A \cap M$ cannot be predense.

If P is ccc, take some dense $D \subseteq P, D \in M$. Find a maximal antichain $A \subseteq D$. Using elementarity, we can assume $A \in M$. Since A is countable and $A \in M$ necessarily $A \subseteq M \subseteq D \cap M$ and A is obviously predense. \square

2.72 HOMOGENEITY AND THE KATOWICE PROBLEM.

2.73 DEFINITION. Homogeneity. An ordering (P, \leq) is *homogeneous* if for all $p, q \in P$ the suborders $(\leftarrow, p]$ and $(\leftarrow, q]$ are isomorphic. A Boolean algebra is homogeneous if (B^+, \leq) is a homogeneous ordering.

2.74 FACT. *An infinite Boolean algebra is homogeneous iff one of the following equivalent conditions holds:*

- (i) *For all $a \in B^+$ the factor $B \upharpoonright a$ is isomorphic to B .*
- (ii) *For all $a, b \in B \setminus \{0, 1\}$ there is an automorphism h of B such that $h(a) = b$.*
- (iii) *An automorphism of any finite subalgebra of B can be extended to an automorphism of B .*

Also note that an infinite homogeneous Boolean algebra must be atomless.

The Katowice problem: uncountability versus countability from the point of view of finiteness. There are several formulations which are equivalent in ZFC:

- (i) Is it consistent that the Boolean algebras $\mathcal{P}(\omega)/_{[\omega]^{<\omega}}$ and $\mathcal{P}(\omega_1)/_{[\omega_1]^{<\omega}}$ are isomorphic?
- (ii) Is it consistent that the topological spaces ω^* and ω_1^* are homeomorphic?
- (iii) Is it consistent that $\mathcal{P}(\omega_1)/_{[\omega_1]^{<\omega}}$ is homogeneous?
- (iv) Is it consistent that there is a uniform ultrafilter \mathcal{U}_0 on ω and \mathcal{U}_1 a uniform ultrafilter on ω_1 such that $(\mathcal{U}_0, \subseteq^*)$ and $(\mathcal{U}_1, \subseteq^*)$ are isomorphic orderings?

This is a longstanding open problem. However the following is known

2.75 FACT. *If ω^* is homeomorphic to ω_1^* then $2^\omega = 2^{\omega_1}$ and $\mathfrak{d} = \omega_1$.*

For more information see [Nyi07].