The tree property at $\aleph_{\omega+2}$ with a finite gap

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Introduction

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5. Some combinations of these ... (such as 1+4, 2+4, 1+3) are consistent, some open (such as 2+3, 3+4, 1+2).

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Goal

In this talk we will focus on the combination 1+3, i.e.

- Have $2^{\aleph_\omega}$ large + have the tree property at $\aleph_{\omega+2}$.

We will try to refine the known results by getting $2^{\aleph_\omega}$ as large as possible.

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  - If GCH then $\neg\mathsf{TP}(\kappa^{++})$ for all $\kappa \geq \omega$.
  - $\mathsf{TP}(\kappa^{++})$ then $2^\kappa > \kappa^+$.

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- Recently, gap 2 was also proved (by another method) by Cummings, Friedman, Magidor, Rinot, Sinapova (starting with a supercompact cardinal).

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- It is open whether $2^{\aleph_\omega}$ can be greater or equal to $\aleph_{\omega_1}$ (pcf conjecture implies no).

The tree property at $\aleph_{\omega+2}$ with gap 3

We show a theorem for gap 3 (can be generalized to a finite gap):

Theorem (Friedman, Honzik, S. (2017))

Suppose there is κ which is $H(\lambda^+)$-hypermeasurable where λ is the least weakly compact above κ. Then there is a forcing extension where the following hold:

1. $\kappa = \aleph_\omega$ *is strong limit and* $2^{\aleph_\omega} = \aleph_{\omega+3}$.
2. $\mathrm{TP}(\aleph_{\omega+2})$.

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- In $V[\mathbb{M}]$, κ is still measurable witnessed by some measure U and one can construct a guiding generic G^g and define the Prikry forcing with collapses $\mathbb{P}(U, G^g)$.
- One can show that over V, $\mathbb{M} * \mathbb{P}(U, G^g)$ forces $\kappa = \aleph_\omega$, $\lambda = \aleph_{\omega+2}$, and $2^{\aleph_\omega} = \aleph_{\omega+3}$.

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  2. $\mathbb{T}$ is κ^+-closed.

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- One can define a restriction $\mathbb{M} * \mathbb{P}(U, G^g)|\alpha$ for suitable α's, $\alpha < \lambda$, and carry out the product analysis of the tail forcing in the generic extension by $\mathbb{M} * \mathbb{P}(U, G^g)|\alpha$.

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- The last assumption is used to obtain a contradiction using properties such as the κ^+-cc of $\mathbb{C}^2$ and the product analysis $\mathbb{C} \times \mathbb{T}$ over a suitable quotient.

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