Laver-like indestructibility for measurable cardinals

Radek Honzik

Department of Logic
Charles University

logika.ff.cuni.cz/radek

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Remark: In a different terminology, "the strongness" of κ is measured by how much of the V hierarchy is included in M: κ is $\kappa + \xi$-strong if $V_{\kappa+\xi} \subseteq M$.

Laver's result

Theorem (Laver (1977))

Suppose κ is supercompact. Then there is a reverse-Easton forcing iteration P_κ of length κ, size κ and satisfying κ-cc such that for any $\dot{Q}$ if $P_\kappa \Vdash$ "$\dot{Q}$ is κ-directed closed," then

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- Notice that κ is indestructible by a proper class of forcings while the "Laver preparation" P_κ is a set forcing.
- One cannot replace *κ-directed closed* by *κ-closed*: the forcing adding a κ-Kurepa tree is a counterexample.

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2. As an interesting result in its own right: we can find a forcing P (P_κ in Laver's result) which makes the supercompactness of κ immune to further forcings from a specific (and as large as possible) class of forcings. [A part of the larger program of "undoing" the power of forcing – by forcing.]

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This talk prefers to look at this topic from the point (1).

Related results

Gitik and Shelah (1989) formulated an analog of Laver's result for strong cardinals (with respect to κ^+-weakly closed Prikry-style forcing notions), and Hamkins and others for some other large cardinals (Hamkins (2008) for a strongly unfoldable κ with respect to Cohen forcing at κ of arbitrary length, etc.).

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We will discuss in this talk the indestructibility of measurability of κ with respect to Cohen forcing at κ of some bounded length when we start with κ being a $H(\mu)$-hypermeasurable for some regular $\mu > \kappa$.[1]

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Related results

Remark. One can consider indestructibility of other "desirable" properties φ. We just mention one area in this context: suppose $\varphi(\kappa)$ is "κ is regular and greater or equal to $\aleph_2$ and has the tree property".

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Spencer (2012) showed a limited indestructibility of the tree property at $\aleph_2$ with respect to Cohen forcing at ω of an arbitrary length over the Mitchell model.

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1. The so called "Laver functions". They are available for instance for supercompact and strong cardinals.
2. Sum of forcing notions ("lottery sum" of Hamkins).

Recall that if $\mathbb{R}$ is a set of forcing notions, then the sum $\bigoplus \mathbb{R}$ is defined as follows: the conditions are of the form (R, p) where $R \in \mathbb{R}$ and $p \in R$, the ordering is $(R, p) \leq (S, q)$ iff $R = S$ and $p \leq_R q$, and we add an artificial greatest condition 1 which is greater than all the (R, p)'s.

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Assume GCH holds in the ground model V. If κ is $H(\lambda)$-hypermeasurable for some regular $\lambda > \kappa^+$, then in some cofinality-preserving generic extension V^* of V, Cohen forcing $\mathrm{Add}(\kappa, \lambda)$ *yields the measurability of κ in* $V^*[\mathrm{Add}(\kappa, \lambda)]$.

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We will show an outline of the proof and indicate some generalisations.

Initial embeddings

- Our V^* is of the form $V^1[P_\kappa]$, where V^1 is an extension of V be a preparation forcing P^1 of size κ (an iteration of length κ) which preserves the hypermeasurability of κ, and P_κ is a standard reverse-Easton iteration. I.e. Starting with V, we define a forcing $P^1 * P_\kappa$ which will achieve the indestructibility under a specific Cohen forcing.

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- Let $j : V \to M$ witness the $H(\lambda)$-hypermeasurability of κ. If U the normal measure derived from j,[2] let $i : V \to N$ be the derived ultrapower via U.

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Fast function forcing

- Using the fast-function forcing of Woodin, we can assume that there is $f : \kappa \to \kappa$ in V such that $j(f)(\kappa) = \lambda$. Let us denote $f(\alpha)$ by λ_α; let $C(f)$ denote the closed unbounded set of the closure points of f: if $\alpha \in C(f)$, then for all $\beta < \alpha$, $f(\beta) < \alpha$.

Definition of P^1

- P^1 is $\langle (P^1_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa, \alpha \text{ is measurable}, \alpha \in C(f) \rangle * \dot{Q}_\kappa$, with the Easton-support, where

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  - there exists in $V[P^1_\alpha]$ a normal measure U_α on α such that the derived ultrapower embedding i_α satisfies

$$i_\alpha : V[P^1_\alpha] \to N_\alpha[i(P^1_\alpha)]$$

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One can show that P^1 preserves cofinalities.

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- If U^1 is the normal measure derived from j^1, and $i^1 : V^1 \to N^1$ is the ultrapower embedding for U^1, then in V^1 there is g which is $i^1(P)$-generic over N^1, where $P = \mathrm{Add}(\kappa, \lambda)^{V^1}$. i^1 restricted to V is the original i.

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The key ingredient is the existence of the filer g in V^1.

Definition of P_κ

- Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is measurable}, \alpha \in C(f) \rangle,$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha, \lambda_\alpha)$, and λ_α equals $f(\alpha)$.

The end of the proof

The proof finishes with the usual surgery argument with the following lemma due to Woodin (or folklore) which allows us to use the generic filter g added in V^1 (for the i^1-image of $\mathrm{Add}(\kappa,\lambda)^{V^1}$) in the model $V^1[P_\kappa]$.

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Lemma. *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa}=\kappa$. Then for any λ, the term forcing $Q_\lambda = \mathrm{Add}(\kappa,\lambda)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\lambda)$.*

Some generalisations

- It is not hard see that in $V^1[P_\kappa] = V^*$, κ is actually no longer measurable: its measurability is resurrected by $\mathrm{Add}(\kappa, \lambda)$. To ensure measurability of κ in V^*, one may use lottery sum again, and prove for instance the following:

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  Theorem. *Suppose $\lambda = \kappa^{+n}$, for some $1 < n < \omega$, in the argument above. Then one can modify the definition of P_κ (with the same P^1) so that κ is measurable in V^*, and its measurability is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 < \alpha \leq \kappa^{+n}$.*

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  Hint: Set $\dot{Q}_\alpha$ in P_κ to be equal to $\bigoplus \mathbb{S}$, where $\mathbb{S} = \{1\} \cup \{\mathrm{Add}(\alpha, \alpha^{+k}) \,|\, 0 \le k \le n\}$.

Open questions

The following can probably be obtained using the ideas in the proof:

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- GCH. Suppose $\lambda > \kappa^+$ is regular and κ is $H(\lambda)$-hypermeasurable. Is there a forcing $P^1 * P_\kappa$ which forces that the measurability of κ is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 < \alpha \leq \lambda$?