Answer to a question of Rosłanowski and Shelah

Small-large subgroups of locally compact groups

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Introduction

Small-large subsets

- Orthogonality of $\mathcal{N}$ and $\mathcal{M}$:

$$\exists H \subseteq \mathbb{R},\ H \in \mathcal{N},\ \mathbb{R} \setminus H \in \mathcal{M}.$$

- Obviously H (similarly $\mathbb{R} \setminus H$) cannot be a subgroup:

$$H \leq \mathbb{R}, \quad H \in \mathcal{N}$$

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$$\exists c \notin H$$

H, $(H + c)$ are disjoint co-meager sets.

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Subgroups contained in exactly one ideal

Null but non-meager subgroups:

Theorem (Talagrand, 1980.)

There exist a null but non-meager filter in 2^ω.

Theorem (Rosłanowski-Shelah, 2016.)

There exists a null but non-meager subgroup in 2^ω (and in $\mathbb{R}$).

Corollary

There is no translation invariant Borel hull operation on $\mathcal{N}$.

Definition

$f : \mathcal{N} \to \mathcal{N} \cap \mathcal{B}$ is a translation invariant Borel hull operation on $\mathcal{N}$, if $N \subseteq f(N)$ and $f(N + x) = f(N) + x$ $(\forall N, x)$

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Meager but non-null subgroups in 2^ω (and in $\mathbb{R}$):

Independent:

$$\mathrm{CH} \Rightarrow \mathrm{cof}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) \Rightarrow \exists H \leq 2^\omega \text{ (and } \mathbb{R}\text{)},\ H \in \mathcal{M} \setminus \mathcal{N}$$

$$\mathrm{non}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) \Rightarrow \exists H \leq 2^\omega \text{ (and } \mathbb{R}\text{)},\ H \in \mathcal{M} \setminus \mathcal{N}$$

Remark

This latter generalizes to Polish locally compact groups.

Theorem (Rosłanowski-Shelah, 2016.)

It is consistent with ZFC that every meager subgroup is null in 2^ω (and in $\mathbb{R}$).

This latter theorem also follows from an unpublished result of H. Friedman.

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The general case

Rosłanowski and Shelah asked the following questions:

Question

Does every non-discrete locally compact group contain a null non–meager subgroup?

Question

Is it consistent with ZFC that in every locally compact group every meager subgroup is null?

The general case

Definition (left Haar measure on a locally compact group)

If $\mu_L : \mathcal{B}(G) \to [0, \infty]$ is a Borel measure which is

- left-invariant, i.e. $\mu_L(B) = \mu_L(gB)$,
- inner regular, i.e. $\mu_L(B) = \sup\{\mu_L(K) : \ K \subseteq B, K \text{ is compact}\}$,
- $\mu_L(U) > 0$ for $U \neq \emptyset$ open, $\mu_L(K) < \infty$ for K compact,

then μ_L is a left Haar measure of G. Let $\overline{\mu}_L$ denote its completion.

Remark

For an arbitrary locally compact group left and right Haar measures always exist, and both are unique up to a positive multiplicative constant.

Remark

Recall that the ideal of null sets wrt. $\overline{\mu}_L$ coincides with the ideal of null sets wrt. $\overline{\mu}_R$, i.e. we can speak about null sets:

$$\mathcal{N} = \mathcal{N}_R = \mathcal{N}_L$$

Answer

For both questions we have affirmative answers:

Theorem (M.P. 2016.)

If G is an arbitrary non-discrete locally compact group, then there exists a null but non-meager subgroup in G.

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In the Cohen model in every locally compact group every meager subgroup is null.

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Null but non-meager subgroups in 2^ω

Theorem

There exists a null but non-meager subgroup of 2^ω.

Proof (Rosłanowski-Shelah)

Fix a non-principal ultrafilter $\mathcal{U}$ on ω. Partition ω into disjoint intervals:

$$I_j = [j^2, (j+1)^2),$$

and let

$$H = \{x \in 2^\omega : \ \{j : \ x_{|I_j} \equiv \underline{0}\} \in \mathcal{U}\},$$

i.e. those sequences that are constant 0 on $\mathcal{U}$-almost every I_j-s.

Meager but non-null subgroups in 2^ω

Theorem (H. Friedman)

The following holds in the Cohen model: Assume that $F \subseteq 2^\omega \times 2^\omega$ is an F_σ-set which contains a non-null rectangle

$$C \times D \subseteq F.$$

Then F contains a non-null measurable rectangle $A \times B$.

Corollary (Fremlin, Shelah)

In the Cohen model every meager subgroup of 2^ω is null.

Proof.

Let the meager subgroup $H \leq 2^\omega$ be covered by an F_σ meager set $S \subseteq 2^\omega$:

$$H \subseteq S$$

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Proof.

$$H \subseteq S$$

- Let $f : 2^\omega \times 2^\omega \to 2^\omega$ denote the group operation.
- Then $f^{-1}(S) \subseteq 2^\omega \times 2^\omega$ is an F_σ-set, containing $H \times H$, since H is a subgroup.
- If $H \times H$ were non-null, then using Friedman's theorem there would be a measurable non-null rectangle

$$A \times B \subseteq f^{-1}(S).$$

- But using Steinhaus's theorem, $f(A, B) = A + B \subseteq S$ contains a nonempty open set, contradicting that S is meager.

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Null but non-meager subgroups in locally compact groups

Theorem (M.P. 2016.)

If G is a non-discrete locally compact group, then there exists a null but non-meager subgroup in G.

Our proof splits into the following three steps:

- Step 1: For an arbitrary G, find a compact normal subgroup $K \triangleleft G$ such that G/K is a Polish Lie group, or a Polish profinite group.
- Step 2: Assuming that there exists a null but non-meager subgroup in G/K, verify that its pull-back under the quotient mapping is also null and non-meager in G.
- Step 3: Construction of null but non-meager subgroups in Polish Lie groups, and Polish profinite groups.

Null but non-meager subgroups in locally compact groups

For Steps 1 & 2 our main tool is the following:

Theorem (Gleason-Yamabe)

Let G be an arbitrary locally compact group. Then there exists an open subgroup $G' \leq G$ such that for each neighborhood U of the identity there is a compact normal subgroup $K \lhd G'$ inside U, and G'/K is a Lie group.

With a slight modification we obtain a technical lemma:

Lemma

Let G be an arbitrary locally compact group. Then there exists an open subgroup $G' \leq G$ such that for each neighborhood U of the identity there is a compact normal subgroup $K \lhd G'$ inside U, and G'/K is a Polish Lie group.

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Step 2

The following will be the key for Step 2.

Lemma

Let $K \lhd G$ be a compact normal subgroup such that G/K is Polish. Let $R \subseteq G$ be a co-meager set. Then there exists a compact normal subgroup $K' \subseteq K$, where G/K' is Polish, and a co-meager set $R' \subseteq G/K'$ such that

$$\pi^{-1}(R') \subseteq R$$

where $\pi : G \to G/K'$ denotes the canonical projection.

Let $H/K \leq G/K$ be a non-meager subgroup.
If the pull-back H were meager in G, then letting $R = G \setminus H$, and applying the above Lemma, there would exist a compact normal subgroup $K' \lhd G$, $K' \subseteq K$, such that $H/K' \leq G/K'$ is meager.
Now since $H/K \leq G/K$ is non-meager, its preimage H/K' under the projection $\psi : G/K' \to (G/K')/(K/K') = G/K$ is also non-meager since ψ is a continuous open mapping between Polish spaces, a contradiction.

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Meager but non-null subgroups, Step 1

Theorem (M.P. 2016.)

In the Cohen model in every locally compact group every meager subgroup is null.

- Step 1: Friedman's theorem holds for Polish locally compact groups.
- Step 2: Assuming that our theorem holds for Polish locally compact groups, it holds for all locally compact groups.

Step 1: Fixing a locally compact Polish group G, using the Measure Isomorphism Theorem there exist F_σ co-null sets $C \subseteq 2^\omega$, $K \subseteq G$, and a bijection

$$f : C \to K$$

such that

$$A \subseteq C \text{ is null} \iff f(A) \subseteq K \text{ is null},$$

$$A \subseteq C \text{ is } F_\sigma \iff f(A) \subseteq K \text{ is } F_\sigma.$$

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Lemma

Let G be a locally compact group which is an inverse limit of Polish Lie groups. Let $R \subseteq G$ be a co-meager set. Then there exists a compact normal subgroup K, where G/K is Polish, and a co-meager set $R' \subseteq G/K$ such that

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Step 2: Assuming that $H \leq G$ is a meager subgroup, let $R = G \setminus H$.

- Then applying the above Lemma, $\pi^{-1}(R') \cap H = \emptyset$, thus $\pi(H) \cap R' = \emptyset$, i.e. $\pi(H) \leq G/K$ is a meager subgroup. Thus is null, since the theorem holds for Polish groups.
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Question

What can we say about the case of non locally compact Polish groups, replacing null sets wrt. the Haar measure with sets in the Haar-null ideal (in the sense of Christensen), i.e. subgroups in $\mathcal{HN} \setminus \mathcal{M}$, $\mathcal{M} \setminus \mathcal{HN}$*?*

Definition

Let G be a Polish group. Then a set $A \subseteq G$ is Haar-null (in the sense of Christensen) if there is a Borel probability measure μ on G, and a Borel set $B \supseteq A$, such that for every $g, h \in G$ $\mu(gBh) = 0$

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(CH) There is a subgroup $H \leq \mathbb{R}$, $H \in \mathcal{M} \setminus \mathcal{N}$*, but what about Polish locally compact groups?*

Question (Filipczak-Rosłanowski-Shelah)

Is it consistent that there is a translation invariant Borel hull operation on $\mathcal{M}$*?*

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