The Nikodym property of Boolean algebras and cardinal invariants of the continuum

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Let’s start with measures

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If $K$ is a compact Hausdorff space, then $C(K)$ denotes the Banach space of real-valued continuous functions on $K$. The dual space $C(K)^*$ is the space of all bounded regular Borel measures on $K$. 

Question

Let $\langle \mu_n : n \in \omega \rangle$ be a sequence of measures on a Boolean algebra $\mathcal{A}$. Assume that $\lim_{n \to \infty} \mu_n(\mathcal{A}) = 0$ for every $\mathcal{A} \in \mathcal{A}$. Does it follow that $\lim_{n \to \infty} \int_{K_{\mathcal{A}}} f \, d\mu_n = 0$ for every $f \in C(K_{\mathcal{A}})$?
A **measure** \( \mu \) on a Boolean algebra \( \mathcal{A} \) is a signed real-valued finitely additive function of finite variation. If \( \mu \) is a measure on \( \mathcal{A} \), then \( \mu \) extends uniquely to a regular Borel (\( \sigma \)-additive) measure \( \mu \) on the Stone space \( K_{\mathcal{A}} \) of \( \mathcal{A} \) (with the same variation).

If \( K \) is a compact Hausdorff space, then \( C(K) \) denotes the Banach space of real-valued continuous functions on \( K \). The dual space \( C(K)^* \) is the space of all bounded regular Borel measures on \( K \).

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Let \( \langle \mu_n : n \in \omega \rangle \) be a sequence of measures on a Boolean algebra \( \mathcal{A} \). Assume that \( \lim_{n \to \infty} \mu_n(A) = 0 \) for every \( A \in \mathcal{A} \).
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Let $\langle \mu_n : n \in \omega \rangle$ be a sequence of measures on a Boolean algebra $\mathcal{A}$. Assume that $\lim_{n \to \infty} \mu_n(A) = 0$ for every $A \in \mathcal{A}$. Does it follow that

$$\lim_{n \to \infty} \int_{K_{\mathcal{A}}} f \, d\mu_n = 0$$

for every $f \in C(K_{\mathcal{A}})$?
A sequence of measures $\langle \mu_n : n \in \omega \rangle$ on $\mathcal{A}$ is

- **pointwise convergent** if $\mu_n(A) \to 0$ for every $A \in \mathcal{A},$

Fact

Let $\mathcal{A}$ be a Boolean algebra. TFAE:

- every pointwise convergent sequence of measures on $\mathcal{A}$ is weak* convergent,
- every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

The question

Let $\langle \mu_n : n \in \omega \rangle$ be a pointwise bounded sequence of measures on a Boolean algebra $\mathcal{A}$. Is $\langle \mu_n : n \in \omega \rangle$ uniformly bounded?
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A sequence of measures \( \langle \mu_n : n \in \omega \rangle \) on \( \mathcal{A} \) is

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Let \( \langle \mu_n : n \in \omega \rangle \) be a pointwise bounded sequence of measures on a Boolean algebra \( \mathcal{A} \). Is \( \langle \mu_n : n \in \omega \rangle \) uniformly bounded?
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- **pointwise convergent** if $\mu_n(A) \to 0$ for every $A \in \mathcal{A}$,
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Let $\langle \mu_n : n \in \omega \rangle$ be a pointwise bounded sequence of measures on a Boolean algebra $\mathcal{A}$. Is $\langle \mu_n : n \in \omega \rangle$ uniformly bounded?
Theorem (Nikodym’s Uniform Boundedness Principle ‘30)

If $\mathcal{A}$ is a $\sigma$-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.
Nikodym’s UBP

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A striking improvement of the UBP!

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A sequence $\langle \mu_n : n \in \omega \rangle$ on $A$ is anti-Nikodym if it is pointwise bounded on $A$ but not uniformly bounded.
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**Theorem (Nikodym’s Uniform Boundedness Principle ’30)**

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**Definition**

An infinite Boolean algebra $\mathcal{A}$ has the Nikodym property (N) if there are no anti-Nikodym sequences on $\mathcal{A}$.
The Nikodym Property

Notable examples

- $\sigma$-algebras (Nikodym ’30),
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However, if the Stone space $K_A$ of $A$ has a convergent sequence, then $A$ does not have (N):

if $x_n \to x$, then put $\mu_n = n(\delta_{x_n} - \delta_x)$
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All the notable examples are of cardinality at least $\mathfrak{c}$!
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**Question**

Is there an infinite Boolean algebra with (N) and cardinality less than $c$?
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The Nikodym Number

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**The Nikodym number**
\[ n = \min\{|A| : \text{infinite } A \text{ has (N)}\}. \]

If \( |A| = \omega \), then \( K_A \subseteq 2^\omega \), so \( A \) does not have (N). Thus:

\[ \omega_1 \leq n \leq c. \]
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**Theorem (Booth ’74)**

$s = \min \{ w(K) : K \text{ compact not sequentially compact} \}$. 

**Theorem (Geschke ’06)**

Let $K$ be infinite compact and such that $w(K) < \text{cov}(M)$. Then, $K$ is either scattered or $K$ contains a perfect subset $L$ with a $G^\delta$-point $x \in L$. In both cases, $K$ contains a convergent sequence.

**Corollary**

$max \{ s, \text{cov}(M) \} \leq \eta$. 

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**Corollary**

$$\max(s, \text{cov}(\mathcal{M})) \leq \eta.$$
Lower bounds for \( n \)

**Proposition**

\[ b \leq n. \]
## Lower bounds for $n$

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### Lower bounds for $n$

#### Proposition

$$ b \leq n. $$

#### Corollary

- $\max (b, s, \text{cov}(M)) \leq n.$
- *Under MA(ctl), $n = c.$*

There is no ZFC inequality between any of $b$, $s$ and $\text{cov}(M)$.

#### Question

$$ d \leq n? $$
Upper bounds for $n$?

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$Fr(\omega) \subseteq \mathcal{A}$
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$\exists Fr(\omega) \subseteq \mathcal{B} \subseteq Fr(\omega)$ with (N) and $|\mathcal{B}| = \mathfrak{n}$. 
Let’s prove Nikodym’s UBP!

**Theorem (Nikodym’s Uniform Boundedness Principle ’30)**

If $\mathcal{A}$ is a $\sigma$-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

**A sketch of the proof**

Let $\mathcal{A}$ be a $\sigma$-complete Boolean algebra. Assume $\mathcal{A}$ does not have (N) — there exists anti-Nikodym $\langle \mu_n : n \in \omega \rangle$ on $\mathcal{A}$. 

Using anti-Nikodymness of $\langle \mu_n : n \in \omega \rangle$ construct a special antichain $\langle a_k : k \in \omega \rangle$ in $\mathcal{A}$ ...

Using specialness of $\langle a_k : k \in \omega \rangle$ obtain a subantichain $\langle a_i : i \in A \rangle$ ($A \in \mathcal{P}(\omega)$) such that:

$$\sup_{k \in A} |\mu_k(\bigvee_{i \in A} a_i) | = \infty.$$ 

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A contradiction!
Let $\kappa$ be a cardinal number. We say that a Boolean algebra $\mathcal{A}$ has the $\kappa$-anti-Nikodym property if there exists a family
\[
\{ \langle a_n^\gamma \in \mathcal{A} : n \in \omega \rangle : \gamma < \kappa \}
\]
of $\kappa$ many antichains in $\mathcal{A}$ with the following property:

for every anti-Nikodym sequence of real-valued measures $\langle \mu_n : n \in \omega \rangle$ on $\mathcal{A}$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n_k : k \in \omega \rangle$ of natural numbers such that for every $k \in \omega$ the following inequality is satisfied:

\[
|\mu_{n_k} (a_k^\gamma)| > \sum_{i=0}^{k-1} |\mu_{n_k} (a_i^\gamma)| + k + 1.
\]
Definition

Let $\kappa$ be a cardinal number. We say that a Boolean algebra $\mathcal{A}$ has the $\kappa$-anti-Nikodym property if there exists a family $\{\langle a_\gamma^n \in \mathcal{A} : n \in \omega \rangle : \gamma < \kappa\}$ of $\kappa$ many antichains in $\mathcal{A}$ with the following property:

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The anti-Nikodym number $n_a$

$$n_a = \min \{ \kappa : \text{every ctbl } \mathcal{A} \text{ has } \kappa\text{-anti-Nikodym property} \}.$$
Definition

Given $\mathcal{F} \subseteq [\omega]^\omega$, an antichain $\langle a_n : n \in \omega \rangle$ in $\mathcal{A}$ is $\mathcal{F}$-complete in $\mathcal{A}$ if $\bigvee_{n \in \mathcal{A}} a_n \in \mathcal{A}$ for every $\mathcal{A} \in \mathcal{F}$. 
Definition

Given $\mathcal{F} \subseteq [\omega]^\omega$, an antichain $\langle a_n : n \in \omega \rangle$ in $\mathcal{A}$ is $\mathcal{F}$-complete in $\mathcal{A}$ if $\bigvee_{n \in A} a_n \in \mathcal{A}$ for every $A \in \mathcal{F}$.

$\mathcal{A}$ is $\sigma$-complete iff every antichain in $\mathcal{A}$ is $[\omega]^\omega$-complete.
Two auxiliary numbers

Definition

A family $\mathcal{F} \subseteq [\omega]^{\omega}$ is Nikodym extracting if for every algebra $\mathcal{A}$ the following condition holds:

for every sequence $\langle \mu_n : n \in \omega \rangle$ of positive measures on $\mathcal{A}$ and every $\mathcal{F}$-complete antichain $\langle a_n \in \mathcal{A} : n \in \omega \rangle$ in $\mathcal{A}$, there is $A \in \mathcal{F}$ such that the following inequality is satisfied:

$$\mu_n \left( \bigvee_{k \in A, k > n} a_k \right) < 1$$

for every $n \in A$. 

Darst '67: $\mathcal{F} \subseteq [\omega]^{\omega}$ is Nikodym extracting.

The Nikodym extracting number $n_{\text{en}} = \min \{|F| : F \subseteq [\omega]^{\omega} \text{ is Nikodym extracting}\}$. 
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A family $\mathcal{F} \subseteq [\omega]^\omega$ is **Nikodym extracting** if for every algebra $\mathcal{A}$ the following condition holds:

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**The Nikodym extracting number $n_e$**

$$n_e = \min \{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is Nikodym extracting}\}.$$
Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}([\kappa]^\omega)$ (then $\text{cf}(\kappa) > \omega!$).
The construction

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Fix a Nikodym extracting family \( \mathcal{G} \subseteq [\omega]^\omega \), \( |\mathcal{G}| = n_e \).
The construction

Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}(\kappa^\omega)$ (then $\text{cf}(\kappa) > \omega$). Fix a Nikodym extracting family $G \subseteq \omega^\omega$, $|G| = n_e$.

- Start with some $B_0 \subseteq \wp(\kappa)$, $|B_0| = \kappa$. 
Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}(\kappa^\omega)$ (then $\text{cf}(\kappa) > \omega$!). Fix a Nikodym extracting family $G \subseteq [\omega]^\omega$, $|G| = n_e$.

- Start with some $B_0 \subseteq \wp(\kappa)$, $|B_0| = \kappa$.
- On a successor step:
  1. take cofinal $\mathcal{F} \subseteq [B_\eta]^\omega$, $|\mathcal{F}| = \kappa$;

On a limit step take the union of preceding algebras. Continue until $A = B_\omega$ is obtained. $A$ has the Nikodym property and cardinality $\kappa$. 
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  2. for every $A \in F$ take $\{\langle a_\gamma^n : n \in \omega \rangle : \gamma < n_a\}$ witnessing $n_a$-anti-Nikodymness;
Let \( \kappa \geq \max(n_a, n_e) \) be such that \( \kappa = \text{cof}(\kappa^\omega) \) (then \( \text{cf}(\kappa) > \omega! \)). Fix a Nikodym extracting family \( G \subseteq \omega^\omega, |G| = n_e \).

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  2. for every \( A \in F \) take \( \{ \langle a_\gamma^n : n \in \omega \rangle : \gamma < n_a \} \) witnessing \( n_a \)-anti-Nikodymness;
  3. put \( b_\gamma^A = \bigvee_{n \in A} a_\gamma^n \) for every \( A \in G \) and \( \gamma < n_a \);
The construction

Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}([\kappa]^\omega)$ (then $\text{cf}(\kappa) > \omega!$). Fix a Nikodym extracting family $\mathcal{G} \subseteq [\omega]^\omega$, $|\mathcal{G}| = n_e$.

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  2. for every $A \in \mathcal{F}$ take $\{\langle a_n^\gamma : n \in \omega \rangle : \gamma < n_a\}$ witnessing $n_a$-anti-Nikodymness;
  3. put $b_A^\gamma = \bigvee_{n \in A} a_n^\gamma$ for every $A \in \mathcal{G}$ and $\gamma < n_a$;
  4. put $\Phi(A) = \{b_A^\gamma : A \in \mathcal{G}, \gamma < n_a\}$;

On a limit step take the union of preceding algebras. Continue until $\mathcal{B} = \mathcal{B}_\omega$ is obtained. $\mathcal{A}$ has the Nikodym property and cardinality $\kappa$. 
The construction

Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}([\kappa]^\omega)$ (then $\text{cf}(\kappa) > \omega!$). Fix a Nikodym extracting family $\mathcal{G} \subseteq [\omega]^\omega$, $|\mathcal{G}| = n_e$.

- Start with some $\mathcal{B}_0 \subseteq \wp(\kappa)$, $|\mathcal{B}_0| = \kappa$.
- On a successor step:
  1. take cofinal $\mathcal{F} \subseteq [\mathcal{B}_\eta]^\omega$, $|\mathcal{F}| = \kappa$;
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  5. $\mathcal{B}_{\eta+1}$ is generated by $\mathcal{B}_\eta \cup \bigcup_{A \in \mathcal{F}} \Phi(A)$.
Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}([\kappa]^{\omega})$ (then $\text{cf}(\kappa) > \omega!$). Fix a Nikodym extracting family $G \subseteq [\omega]^{\omega}$, $|G| = n_e$.

- Start with some $B_0 \subseteq \wp(\kappa)$, $|B_0| = \kappa$.
- On a successor step:
  1. take cofinal $F \subseteq [B_\eta]^{\omega}$, $|F| = \kappa$;
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- On a limit step take the union of preceding algebras.
- Continue until $\mathcal{A} = \mathcal{B}_{\omega_1}$ is obtained.
Let $\kappa \geq \max(n_a, n_e)$ be such that $\kappa = \text{cof}(\kappa^\omega)$ (then $\text{cf}(\kappa) > \omega!$). Fix a Nikodym extracting family $G \subseteq [\omega]^\omega$, $|G| = n_e$.

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  5. $B_{\eta+1}$ is generated by $B_\eta \cup \bigcup_{A \in F} \Phi(A)$.
- On a limit step take the union of preceding algebras.
- Continue until $A = B_{\omega_1}$ is obtained.

$A$ has the Nikodym property and cardinality $\kappa$. 

The construction
Theorem

Assume that $\max(n_a, n_e) \leq \kappa$ for a cardinal number $\kappa$ such that $\text{cof}([\kappa]^{\omega}) = \kappa$. Then, there exists a Boolean algebra $A$ with the Nikodym property and of cardinality $\kappa$. 
The anti-Nikodym number $n_a$

$$n_a = \min \{ \kappa : \text{every ctbl } \mathcal{A} \text{ has } \kappa\text{-anti-Nikodym property} \}.$$
The anti-Nikodym number

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Proposition

Let $A, B$ be Boolean algebras and $h : A \rightarrow B$ an epimorphism. Then, $n_a(A) \geq n_a(B)$. 
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Let $\mathcal{A}, \mathcal{B}$ be Boolean algebras and $h : \mathcal{A} \rightarrow \mathcal{B}$ an epimorphism. Then, $n_a(\mathcal{A}) \geq n_a(\mathcal{B}).$

Corollary

For any countable $\mathcal{A}$ we have:

$n_a(FC) \leq n_a(\mathcal{A}) \leq n_a(Fr(\omega)) = n_a.$
The anti-Nikodym number

Proposition

1. $b \leq n_a(FC) \leq \text{cof}(\mathcal{M})$. 
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Proposition

1. $b \leq n_a(FC) \leq \text{cof}(\mathcal{M})$.
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\[ \begin{array}{cccccc}
\text{cov}(\mathcal{N}) & \rightarrow & \text{non}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{N}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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**Definition**

An ultrafilter $\mathcal{U}$ on $\omega$ is **selective** (*Ramsey*) if for every partition $\mathcal{P}$ of $\omega$ disjoint with $\mathcal{U}$ there is $A \in \mathcal{U}$ such that $|A \cap P| \leq 1$ for every $P \in \mathcal{P}$.

Theorem (Kunen '76)

The existence of selective ultrafilters is undecidable in ZFC.

The selective ultrafilter number $u_s$

\[ u_s = \min \{|F| : F \text{ is a basis of a selective ultrafilter} \}. \]
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\[ \text{cov}(\mathcal{M}) \leq n_e \leq \min(\vartheta, u_s). \]
The Nikodym extracting number

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\[ \text{cov}(\mathcal{M}) \leq n_e \leq \min(\mathbb{d}, u_s). \]
Summary

Theorem

1. $b \leq n_a \leq \text{cof}(\mathcal{N})$.
2. $\text{cov}(\mathcal{M}) \leq n_e \leq \min(d, u_s)$.
3. If $\text{cof}([\kappa]^\omega) = \kappa \geq \max(n_a, n_e)$, then $n \leq \kappa$. 
Theorem

1. $b \leq n_a \leq \text{cof}(\mathcal{N})$.
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Consistently, $n < c$. 

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Consequence – cofinality and homomorphism type

**Definition**

\[ \text{cof}(A) = \min\{\kappa : \exists \langle A_\xi : \xi < \kappa \rangle \downarrow A\} \].
Consequence – cofinality and homomorphism type

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\text{cof}(\mathcal{A}) = \min\{\kappa : \exists \langle A_\xi : \xi < \kappa \rangle \uparrow \mathcal{A}\}.
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h(\mathcal{A}) = \min\{|\phi(\mathcal{A})| : \phi \text{ is a homomorphism}\}.
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Consequence – cofinality and homomorphism type

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**Theorem (Koppelberg ’77)**

1. \(\omega \leq \text{cof}(\mathcal{A}) \leq h(\mathcal{A}) \leq c\),
2. (MA) If \(|\mathcal{A}| < c\), then \(\text{cof}(\mathcal{A}) = h(\mathcal{A}) = \omega\).
Consequence – cofinality and homomorphism type

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\[
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\text{cof}(A) & = \min\{\kappa : \exists \langle A_\xi : \xi < \kappa \rangle \uparrow A\}. \\
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**Theorem (Just–Koszmider ’91)**

*In the Sacks model there exists a Boolean algebra \(B\) such that \(|B| = \text{cof}(B) = h(B) = \omega_1\).*
Consequence – cofinality and homomorphism type

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Theorem (Pawlikowski–Ciesielski ’02)

*Assuming \( \text{cof}(\mathcal{N}) = \omega_1 \), there exists a Boolean algebra \( \mathcal{B} \) such that \( |\mathcal{B}| = \text{cof}(\mathcal{B}) = \omega_1 \).*
Theorem (Schachermayer '82)

*If* \( A \) *has the Nikodym property, then* \( \text{cof}(A) > \omega \).*
Theorem (Schachermayer ’82)

\[ \text{If } \mathcal{A} \text{ has the Nikodym property, then } \text{cof}(\mathcal{A}) > \omega. \]

Corollary

Assuming \( \text{cof}(\mathcal{N}) \leq \kappa = \text{cof}([\kappa]^{\omega}) \), there exists a Boolean algebra \( \mathcal{A} \) such that \( |\mathcal{A}| = \kappa \), \( h(\mathcal{A}) \geq \aleph \) and \( \text{cof}(\mathcal{A}) = \omega_1 \).
Consequence – cofinality of Boolean algebras

**Theorem (Schachermayer ’82)**

*If $\mathcal{A}$ has the Nikodym property, then $\text{cof}(\mathcal{A}) > \omega$.***

**Corollary**

Assuming $\text{cof}(\mathcal{N}) \leq \kappa = \text{cof}([\kappa]^{\omega})$, there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$, $h(\mathcal{A}) \geq n$ and $\text{cof}(\mathcal{A}) = \omega_1$.

**An old open question**

Is there a consistent example of a Boolean algebra $\mathcal{B}$ for which $\omega_1 < \text{cof}(\mathcal{B}) < c$?
Definition

An infinite compact Hausdorff space is a *Efimov space* if it contains neither a convergent sequence nor a copy of $\beta\omega$. 
## Consequence – the Efimov problem

### Definition
An infinite compact Hausdorff space is a **Efimov space** if it contains neither a convergent sequence nor a copy of $\beta \omega$.

### The Efimov Problem ’69
Does there exist a Efimov space?
Consequence – the Efimov problem

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Fedorčuk: CH, $\diamondsuit$, $s = \omega_1$ & $c = 2^{\omega_1}$
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### Theorem
*Assuming* $\text{cof}(\mathcal{N}) \leq \kappa = \text{cof}([\kappa]^\omega) < c$, *there exists a Efimov space* $K$ *such that* $w(K) = \kappa$ *and for every infinite closed subset* $L$ *of* $K$ *we have* $w(L) \geq \eta$. 

Thank you for the attention!