The Nikodym property of Boolean algebras and cardinal invariants of the continuum

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Let $\langle \mu_n \colon\ n \in \omega \rangle$ be a sequence of measures on a Boolean algebra $\mathcal{A}$. Assume that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{A}$.

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$$\lim_{n\to\infty} \int_{K_{\mathcal{A}}} f \, d\mu_n = 0 \quad \text{for every } f \in C(K_{\mathcal{A}})?$$

Pointwise boundedness vs. uniform boundedness

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Let $\mathcal{A}$ be a Boolean algebra. TFAE:

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The question

Let $\langle \mu_n \colon\ n \in \omega \rangle$ be a pointwise bounded sequence of measures on a Boolean algebra $\mathcal{A}$. Is $\langle \mu_n \colon\ n \in \omega \rangle$ uniformly bounded?

Nikodym's UBP

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An infinite Boolean algebra $\mathcal{A}$ has *the Nikodym property* (N) if there are no anti-Nikodym sequences on $\mathcal{A}$.

The Nikodym Property

Notable examples

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However, if the Stone space $K_{\mathcal{A}}$ of $\mathcal{A}$ has a convergent sequence, then $\mathcal{A}$ does not have (N):

$$\text{if } x_n \to x, \text{ then put } \mu_n = n(\delta_{x_n} - \delta_x)$$

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All the notable examples are of cardinality at least $\mathfrak{c}$!

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If $|\mathcal{A}| = \omega$, then $K_\mathcal{A} \subseteq 2^\omega$, so $\mathcal{A}$ does not have (N). Thus:

$$\omega_1 \leqslant \mathfrak{n} \leqslant \mathfrak{c}.$$

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Theorem (Geschke '06)

Let K be infinite compact and such that $w(K) < \mathrm{cov}(\mathcal{M})$. Then, K is either scattered or K contains a perfect subset L with a $\mathbb{G}_\delta$-point $x \in L$. In both cases, K contains a convergent sequence.

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Corollary

$\max(\mathfrak{s}, \operatorname{cov}(\mathcal{M})) \leqslant \mathfrak{n}$.

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- $\max(\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{M})) \leqslant \mathfrak{n}$.
- *Under* MA(ctbl), $\mathfrak{n} = \mathfrak{c}$.

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Proposition

$\mathfrak{b} \leqslant \mathfrak{n}$.

Corollary

- $\max(\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{M})) \leqslant \mathfrak{n}$.
- *Under* MA(ctbl), $\mathfrak{n} = \mathfrak{c}$.

There is no ZFC inequality between any of $\mathfrak{b}$, $\mathfrak{s}$ and $\mathrm{cov}(\mathcal{M})$.

Question

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$\exists$ $Fr(\omega) \subseteq \mathcal{B} \subseteq \overline{Fr(\omega)}$ with (N) and $|\mathcal{B}| = \mathfrak{n}$.

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Theorem (Nikodym's Uniform Boundedness Principle '30)

If $\mathcal{A}$ is a σ-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

A sketch of the proof

Let $\mathcal{A}$ be a σ-complete Boolean algebra. Assume $\mathcal{A}$ does not have (N) — there exists anti-Nikodym $\langle \mu_n \colon\ n \in \omega \rangle$ on $\mathcal{A}$.

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A contradiction!

Two auxiliary numbers

Definition

Let κ be a cardinal number. We say that a Boolean algebra $\mathcal{A}$ has *the κ-anti-Nikodym property* if there exists a family $\{\langle a_n^\gamma \in \mathcal{A}\colon\ n \in \omega\rangle\colon\ \gamma < \kappa\}$ of κ many antichains in $\mathcal{A}$ with the following property:

> *for every anti-Nikodym sequence of real-valued measures $\langle \mu_n\colon\ n \in \omega\rangle$ on $\mathcal{A}$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ of natural numbers such that for every $k \in \omega$ the following inequality is satisfied:*
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> $$\left|\mu_{n_k}\left(a_k^\gamma\right)\right| > \sum_{i=0}^{k-1}\left|\mu_{n_k}\left(a_i^\gamma\right)\right| + k + 1.$$

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The anti-Nikodym number $\mathfrak{n}_a$

$\mathfrak{n}_a = \min\{\kappa\colon\ \text{every ctbl } \mathcal{A} \text{ has } \kappa\text{-anti-Nikodym property}\}.$

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Given $\mathcal{F} \subseteq [\omega]^\omega$, an antichain $\langle a_n : \ n \in \omega \rangle$ in $\mathcal{A}$ is *$\mathcal{F}$-complete in $\mathcal{A}$* if $\bigvee_{n \in A} a_n \in \mathcal{A}$ for every $A \in \mathcal{F}$.

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$\mathcal{A}$ is σ-complete iff every antichain in $\mathcal{A}$ is $[\omega]^\omega$-complete.

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A family $\mathcal{F} \subseteq [\omega]^\omega$ is *Nikodym extracting* if for every algebra $\mathcal{A}$ the following condition holds:

> *for every sequence* $\langle \mu_n \colon \ n \in \omega \rangle$ *of positive measures on* $\mathcal{A}$ *and every* $\mathcal{F}$*-complete antichain* $\langle a_n \in \mathcal{A} \colon \ n \in \omega \rangle$ *in* $\mathcal{A}$*, there is* $A \in \mathcal{F}$ *such that the following inequality is satisfied:*
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Darst '67: $[\omega]^\omega$ is Nikodym extracting.

The Nikodym extracting number $\mathfrak{n}_e$

$\mathfrak{n}_e = \min\left\{|\mathcal{F}| \colon \mathcal{F} \subseteq [\omega]^\omega \text{ is Nikodym extracting}\right\}$.

The construction

Let $\kappa \geqslant \max(\mathfrak{n}_a, \mathfrak{n}_e)$ be such that $\kappa = \mathrm{cof}([\kappa]^\omega)$ (then $\mathrm{cf}(\kappa) > \omega!$).

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  2. for every $\mathcal{A} \in \mathcal{F}$ take $\left\{\left\langle a_n^\gamma : \ n \in \omega\right\rangle : \ \gamma < \mathfrak{n}_a\right\}$ witnessing $\mathfrak{n}_a$-anti-Nikodymness;

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  4. put $\Phi(\mathcal{A}) = \left\{b_A^\gamma\colon\ A \in \mathcal{G}, \gamma < \mathfrak{n}_a\right\}$;

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  3. put $b_A^\gamma = \bigvee_{n \in A} a_n^\gamma$ for every $A \in \mathcal{G}$ and $\gamma < \mathfrak{n}_a$;
  4. put $\Phi(\mathcal{A}) = \left\{b_A^\gamma\colon\ A \in \mathcal{G}, \gamma < \mathfrak{n}_a\right\}$;
  5. $\mathcal{B}_{\eta+1}$ is generated by $\mathcal{B}_\eta \cup \bigcup_{\mathcal{A} \in \mathcal{F}} \Phi(\mathcal{A})$.

The construction

Let $\kappa \geqslant \max(\mathfrak{n}_a, \mathfrak{n}_e)$ be such that $\kappa = \mathrm{cof}([\kappa]^\omega)$ (then $\mathrm{cf}(\kappa) > \omega!$).
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$\mathcal{A}$ has the Nikodym property and cardinality κ.

The theorem

Theorem

Assume that $\max(\mathfrak{n}_a, \mathfrak{n}_e) \leqslant \kappa$ *for a cardinal number* κ *such that* $\operatorname{cof}([\kappa]^\omega) = \kappa$. *Then, there exists a Boolean algebra* $\mathcal{A}$ *with the Nikodym property and of cardinality* κ.

The anti-Nikodym number

The anti-Nikodym number $\mathfrak{n}_a$

$\mathfrak{n}_a = \min \left\{ \kappa \colon \ \underline{\text{every}} \text{ ctbl } \mathcal{A} \text{ has } \kappa\text{-anti-Nikodym property} \right\}.$

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Proposition

Let $\mathcal{A}$, $\mathcal{B}$ be Boolean algebras and $h : \mathcal{A} \to \mathcal{B}$ an epimorphism. Then, $\mathfrak{n}_a(\mathcal{A}) \geqslant \mathfrak{n}_a(\mathcal{B})$.

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Corollary

For any countable $\mathcal{A}$ we have:

$$\mathfrak{n}_a(FC) \leqslant \mathfrak{n}_a(\mathcal{A}) \leqslant \mathfrak{n}_a(Fr(\omega)) = \mathfrak{n}_a.$$

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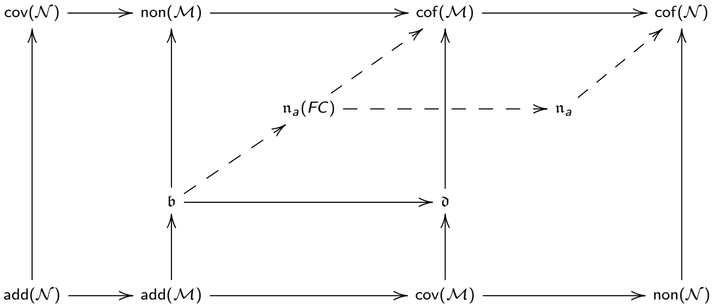

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Definition

An ultrafilter $\mathcal{U}$ on ω is *selective* (*Ramsey*) if for every partition $\mathcal{P}$ of ω disjoint with $\mathcal{U}$ there is $A \in \mathcal{U}$ such that $|A \cap P| \leqslant 1$ for every $P \in \mathcal{P}$.

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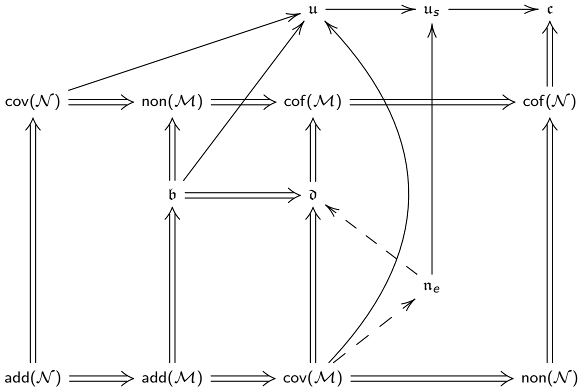

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1. $\mathfrak{b} \leqslant \mathfrak{n}_a \leqslant \operatorname{cof}(\mathcal{N})$.
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Consistently, $\mathfrak{n} < \mathfrak{c}$.

Consequence – cofinality and homomorphism type

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Theorem (Pawlikowski–Ciesielski '02)

Assuming $\mathrm{cof}(\mathcal{N}) = \omega_1$, *there exists a Boolean algebra* $\mathcal{B}$ *such that* $|\mathcal{B}| = \mathrm{cof}(\mathcal{B}) = \omega_1$.

Consequence – cofinality of Boolean algebras

Theorem (Schachermayer '82)

If $\mathcal{A}$ has the Nikodym property, then $\mathrm{cof}(\mathcal{A}) > \omega$.

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Corollary

Assuming $\mathrm{cof}(\mathcal{N}) \leqslant \kappa = \mathrm{cof}([\kappa]^\omega)$, there exists a Boolean algebra $\mathcal{A}$ such that $|\mathcal{A}| = \kappa$, $h(\mathcal{A}) \geqslant \mathfrak{n}$ and $\mathrm{cof}(\mathcal{A}) = \omega_1$.

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An old open question

Is there a consistent example of a Boolean algebra $\mathcal{B}$ for which $\omega_1 < \mathrm{cof}(\mathcal{B}) < \mathfrak{c}$?

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Definition

An infinite compact Hausdorff space is *a Efimov space* if it contains neither a convergent sequence nor a copy of $\beta\omega$.

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Theorem

Assuming $\mathrm{cof}(\mathcal{N}) \leqslant \kappa = \mathrm{cof}([\kappa]^\omega) < \mathfrak{c}$, *there exists a Efimov space* K *such that* $w(K) = \kappa$ *and for every infinite closed subset* L *of* K *we have* $w(L) \geqslant \mathfrak{n}$.

The end

Thank you for the attention!