Cardinal Invariants, Partition Relations, and Generalised Scattered Orders

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Definition

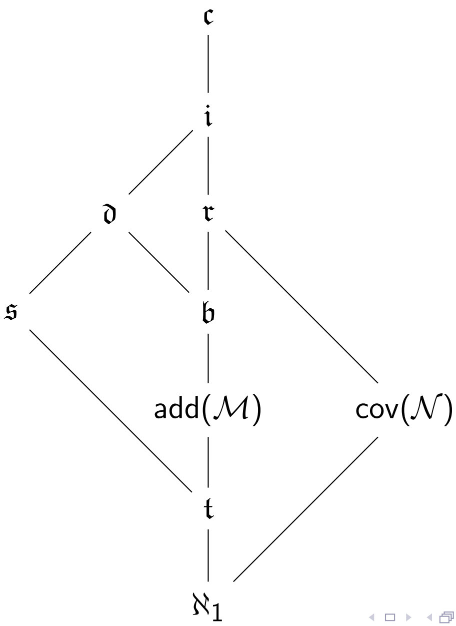
If $f, g \in {}^{\omega}\omega$ then f *dominates* g iff there is a natural number k such that for all natural numbers $n \in \omega \setminus k$ we have $f(n) > g(n)$. A family $F \subset {}^{\omega}\omega$ is *unbounded* iff for all $g \in {}^{\omega}\omega$ there is an $f \in F$ not dominated by g and it is *dominating* if for every $g \in {}^{\omega}\omega$ there is an $f \in F$ which dominates g.

$$\mathfrak{b} := \min \{\#F \mid F \text{ is unbounded.}\}$$
$$\mathfrak{d} := \min \{\#F \mid F \text{ is dominating.}\}$$

Definition

For $x, y \in [\omega]^\omega$ we say that x *splits* y if both $y \cap x$ and $y \setminus x$ are infinite. A family $F \subset [\omega]^\omega$ is called *splitting* if for all $x \in [\omega]^\omega$ there is a $y \in F$ which splits x. It is *reaping* if for all $x \in [\omega]^\omega$ there is a $y \in F$ which is not split by x.

$$\mathfrak{s} := \min \{\#F \mid F \text{ is splitting.}\}$$
$$\mathfrak{r} := \min \{\#F \mid F \text{ is reaping.}\}$$


$\mathfrak{c}$
$\mathfrak{i}$
$\mathfrak{d}$
$\mathfrak{r}$
$\mathfrak{s}$
$\mathfrak{b}$
$\mathrm{add}(\mathcal{M})$
$\mathrm{cov}(\mathcal{N})$
$\mathfrak{t}$
$\aleph_1$

Definition (Ostaszewski 1976)

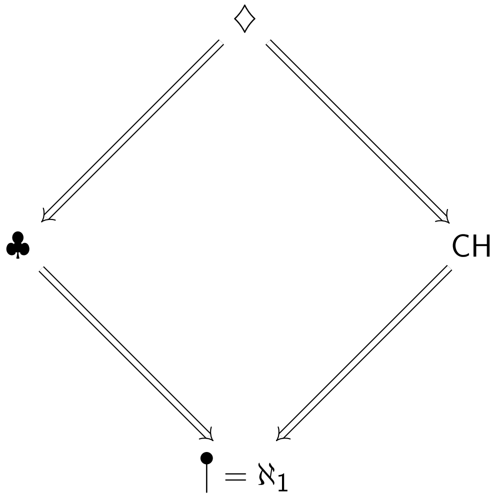
$\clubsuit$ is the statement that there is a sequence $\langle A_\alpha | \alpha < \omega_1 \rangle$ such that for every α, A_α is cofinal in $\omega\alpha$ and for every uncountable $X \subset \omega_1$ there is an $\alpha < \omega_1$ with $A_\alpha \subset X$.

Definition (Broverman, Ginsburg, Kunen, Tall, 1978 and Fuchino, Shelah, Soukup 1997)

⫯ $:= \min \{\#X \mid X \subseteq [\omega_1]^\omega \wedge \forall y \in [\omega_1]^{\omega_1} \exists x \in X : x \subseteq y\}$.

Theorem (Baumgartner 1976)

⫯ $< \mathfrak{c}$ *is consistent.*


$\diamondsuit$
$\clubsuit$
CH
$= \aleph_1$

Theorem (Brendle 2006)

$\mathrm{cov}(\mathcal{N}) = \aleph_2 + \clubsuit$ *is consistent.*

Theorem (Džamonja and Shelah 1999 and Brendle 2006)

$\mathrm{add}(\mathcal{M}) = \aleph_2 + \clubsuit$ *is consistent.*

Theorem (Truss 1984)

If $\Pin = \aleph_1$, *then* $\min\left(\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\right) = \aleph_1$.

Definition

$$\binom{\alpha}{\beta} \longrightarrow \binom{\kappa}{\lambda}_{\xi}$$

means that for every colouring $\chi : \alpha \times \beta \longrightarrow \xi$ there are $A \in [\alpha]^{\kappa}$ and $B \in [\beta]^{\lambda}$ such that χ is constant on $A \times B$.

Definition

$$\binom{\alpha}{\beta} \longrightarrow \begin{bmatrix} \kappa & & \mu \\ & \vee & \\ \lambda & & \nu \end{bmatrix}_{\xi}$$

means that for every colouring $\chi : \alpha \times \beta \longrightarrow \xi$ there are $(A \in [\alpha]^{\kappa}$ and $B \in [\beta]^{\lambda})$ or $(A \in [\alpha]^{\mu}$ and $B \in [\beta]^{\nu})$ such that $\chi\big[A \times B\big] \neq \xi$.

Proposition (Garti, Shelah, 2014)

Suppose $\aleph_0 < \mu < \mathfrak{s}$.

$$\textit{Then } \binom{\mu}{\omega} \longrightarrow \binom{\mu}{\omega}_2 \textit{ iff } \operatorname{cf}(\mu) > \omega.$$

Proposition (Garti, Shelah, 2014)

Suppose $\mathfrak{r} < \mu \leqslant \mathfrak{c}$.

$$\textit{Then } \binom{\mu}{\omega} \longrightarrow \binom{\mu}{\omega}_2 \textit{ whenever } \operatorname{cf}(\mu) > \mathfrak{r}.$$

Problem ([016GS, Problem 2.6])

Is it consistent that $\mathfrak{d} = \aleph_1$ *and* $\binom{\mathfrak{d}}{\omega} \longrightarrow \binom{\mathfrak{d}}{\omega}_2$*?*

Problem ([016GS, Problem 2.10])

Is it consistent that $\mathfrak{i} = \aleph_1$ *and* $\binom{\mathfrak{i}}{\omega} \longrightarrow \binom{\mathfrak{i}}{\omega}_2$*?*

Problem ([016GS, Problem 2.1.])

Suppose $\mathfrak{x}$ *is a nicely defined invariant which satisfies*

$$\mathfrak{x} = \aleph_1 \Rightarrow \binom{\mathfrak{x}}{\omega} \not\rightarrow \binom{\mathfrak{x}}{\omega}_2.$$

Does it follow that $\mathfrak{x} = \mathfrak{c}$*?*

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Proposition (W., 2016)

Suppose that $\mathfrak{b} = \mathfrak{d}$. Then $\binom{\mathfrak{d}}{\omega} \not\to \begin{bmatrix}\mathfrak{b}\\ \omega\end{bmatrix}_{\aleph_0}$

Problem ([O16GS, Problem 2.1.])

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Proposition (W., 2016)

Suppose that $\mathfrak{b} = \mathfrak{d}$. Then $\binom{\mathfrak{d}}{\omega} \not\to \begin{bmatrix}\mathfrak{b}\\ \omega\end{bmatrix}_{\aleph_0}$

Corollary

No, no and probably no.

Theorem (Sierpiński 1933 and Erdős, Hajnal, Rado, 1965)

$$\textit{If } \mathrm{CH}\textit{, then } \binom{\omega_1}{\omega} \not\to \binom{\omega_1}{\omega}_2 .$$

Theorem (Chen, W., 2016)

$$\textit{If } \mathfrak{d} = \aleph_1\textit{, then } \binom{\omega_1}{\omega_1} \not\to \begin{bmatrix} \omega_1 & \vee & \omega \\ \omega & & \omega_1 \end{bmatrix}_{\aleph_0} .$$

Definition

$\alpha \longrightarrow (\beta_{0,0} \vee \cdots \vee \beta_{0,k_0}, \ldots, \beta_{n,0} \vee \cdots \vee \beta_{n,k_n})^i$ means that for every set A of size α and every colouring $\chi : [A]^i \longrightarrow n+1$ there is an $\ell \leqslant n$, an $m \leqslant k_\ell$ and a set $B \subset A$ of size $\beta_{\ell,m}$ such that χ is constant with value ℓ on $[B]^i$.

Theorem (Hajnal, 1971)

If CH, *then* $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Erdős, Hajnal, 1971)

$\omega_1^2 \longrightarrow (\alpha, 3)^2$ *for all* $\alpha < \omega_1^2$.
$\omega_1^2 \longrightarrow (\alpha, n)^2$ *for all* $\alpha < \omega_1\omega$ *and all* $n < \omega$.

Theorem (Takahashi, 1987)

If $⫯ = \aleph_1$, *then* $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Baumgartner, Hajnal, 1987)

$$\omega_1^2 \longrightarrow (\omega_1\omega, 3, 3)^2$$

If CH , *then* $\omega_1^2 \not\longrightarrow (\omega_1\omega, 4)^2$.

Theorem (Larson, 1998)

If $\mathfrak{d} = \aleph_1$, *then* $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Hajnal, 1971)

If CH, *then* $\omega_1 \not\to (\omega_1, \omega + 2)^2$.

Theorem (Todorcevic, 1989)

If $\mathfrak{b} = \aleph_1$, *then* $\omega_1 \not\to (\omega_1, \omega + 2)^2$.

Theorem (Chen, W., 2016)

$\min(\mathfrak{b}, ⫯) = \aleph_1$ *implies* $\omega_1 \not\to (\omega_1, \omega + 2)^2$.

Theorem (Erdős, Hajnal, 1971)

If CH, *then* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (Takahashi, 1987)

If $\max(\mathfrak{d}, \text{⫯}) = \aleph_1$, *then* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (Baumgartner, 1989)

If $\mathrm{MA}(\aleph_1)$, *then* $\omega_1\omega \longrightarrow (\omega_1\omega, n)^2$ *for all natural numbers* n.

Theorem (Larson, 1998)

If $\mathfrak{d} = \aleph_1$, *then* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (W., 2016)

If $\min\left(\mathfrak{d}, \max(\mathfrak{b}, \text{¡})\right) = \aleph_1$, *then* $\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$.

Corollary

If $\max\left(\mathfrak{b}, \min(\mathfrak{d}, \text{¡})\right) = \aleph_1$, *then* $\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$.

Definition

1. A linear order φ is κ-dense if for every $x, y \in \varphi$ with $x < y$ the set $\{z \in \varphi \mid x < z < y\}$ has cardinality κ.
2. A linear order φ is κ-saturated if for every $X, Y \in [\varphi]^{<\kappa}$ with $\forall x \in X \forall y \in Y : x < y$ the set $\{z \in \varphi \mid \forall x \in X \forall y \in Y : x < z < y\}$ is nonempty.

Definition (Džamonja, Thompson, 2006)

Suppose κ is an infinite, regular cardinal, and φ is a linear order type.

1. φ is *κ-scattered* if there is no κ-dense order type τ such that $\tau \leq \varphi$.
2. φ is *weakly κ-scattered* if there is no κ-saturated τ such that $\tau \leq \varphi$.

Remark

Being scattered means being $\aleph_0$-scattered.

Definition

Suppose κ is an infinite, regular cardinal, μ is an infinite cardinal, φ is a linear order type, and P is an order of type φ. φ is $\langle\kappa,\mu\rangle$*-scattered* (resp. *weakly* $\langle\kappa,\mu\rangle$*-scattered*) if there is $\nu<\mu$ and a sequence of suborders $\langle P_\zeta \mid \zeta<\nu\rangle$ of P such that $\mathrm{otp}(P_\zeta)$ is κ-scattered (resp. weakly κ-scattered) for all $\zeta<\nu$ and $\bigcup_{\zeta<\nu} P_\zeta = P$.

Remark

Being σ*-scattered means being* $\langle\aleph_0,\aleph_1\rangle$*-scattered.*

Theorem (Erdős, Milner, 1972)

$\omega^{1+\nu h} \longrightarrow (\omega^{1+\nu}, 2^h)^2$ for all countable ordinals ν and all natural numbers n.

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$\omega^{1+\nu h} \longrightarrow (\omega^{1+\nu}, 2^h)^2$ for all countable ordinals ν and all natural numbers n.

Theorem (Lambie-Hanson, W., 2016)

Suppose $\kappa^{<\kappa} = \kappa$ and φ is a weakly κ-scattered linear order type of size at most κ. Then there is a weakly κ-scattered linear order type τ of size at most κ such that, for all $n < \omega$, $\tau \longrightarrow (\varphi, n)^2$.

Corollary (W. but probably Galvin before)

For all countable scattered linear orders φ there is a countable scattered linear order τ such that for all $n < \omega$ we have $\tau \longrightarrow (\varphi, n)^2$.

Paradox (Milner, Rado, 1965)

For every cardinal κ, every ordinal $\alpha < \kappa^+$ can be writen as a union $\bigcup_{n<\omega} P_n$ such that there is no $n < \omega$ for which P_n has a suborder of type κ^n.

Theorem (Erdős, Hajnal, 1971)

If $\omega_1\omega \not\to (\omega_1\omega, 3)^2$, then $\alpha \not\to (\omega_1^\omega, 3)^2$ for all $\alpha < \omega_2$.

Definition

Let φ be an order type.

1. φ^* denotes the reverse of φ.
2. The product type $\tau\varphi$ is the type of lexicographically ordered set of pairs in $P \times T$ for an order P of type φ and an order T of type τ.
3. Analogously, for an order-type φ and a natural number n the type φ^n denotes the type of lexicographically ordered n-tuples of elements of P for an order P of type φ.

Definition

$(\alpha\alpha^*)^\omega$ is the order type of $S_\alpha := \alpha^{<\omega}$, ordered by $\prec_\alpha$ as follows:

For $s, t \in S_\alpha$ let

$$s \prec t \Leftrightarrow \begin{cases} \ell(s) \text{ is even and } \ell(t) \text{ is odd or} \\ \ell(s) \text{ and } \ell(t) \text{ are both even and } \ell(t) < \ell(s) \text{ or} \\ \ell(s) \text{ and } \ell(t) \text{ are both odd and } \ell(s) < \ell(t) \text{ or} \\ \ell(s) = \ell(t) \text{ is even and } t <_{\text{lex}} s \text{ or} \\ \ell(s) = \ell(t) \text{ is odd and } s <_{\text{lex}} t. \end{cases}$$

Theorem (Lambie-Hanson, W., 2016)

Let κ, μ be infinite regular cardinals such that $\kappa \leq \mu$, and suppose φ is a $\langle \kappa, \max(\aleph_1, \kappa) \rangle$-scattered linear order type of size at most μ. Then every order P of type φ can be written as a union $P = \bigcup_{n<\omega} P_n$ such that there is no $n < \omega$ for which P_n has a suborder of type $\mu^n, (\mu^n)^*$, $(\kappa\kappa^*)^n$, or $(\kappa^*\kappa)^n$.

Corollary

Suppose φ is an σ-scattered linear order type of size at most $\aleph_1$. Then every order P of type φ can be written as a union $P = \bigcup_{n<\omega} P_n$ such that there is no $n < \omega$ for which P_n has a suborder of type ω_1^n, $(\omega_1^n)^*$ or $(\omega\omega^*)^n$.

Theorem (W., 2016)

Assume $\omega_1\omega \not\to (\omega_1\omega, 3)^2$, *let* τ *be a* σ*-scattered linear order type of size at most* $\aleph_1$, *and let* ρ *be an order type such that* $(\omega\omega^*)^n \leq \rho$ *for all natural numbers n. Then*

$$\tau \not\to \left(\omega_1^\omega \vee (\omega_1^\omega)^* \vee \omega_1\rho \vee \omega_1^*\rho \vee \rho\omega_1 \vee \rho\omega_1^*, 3\right)^2.$$

Corollary

Assume $\max\left(\mathfrak{b}, \min(\mathfrak{d}, \text{⫯})\right) = \aleph_1$, *and let* τ *be a* σ*-scattered linear order type of size at most* $\aleph_1$. *Then*

$$\tau \not\to \left(\omega_1^\omega \vee (\omega_1^\omega)^* \vee \omega_1(\omega\omega^*)^\omega \vee \omega_1^*(\omega\omega^*)^\omega \vee (\omega\omega^*)^\omega\omega_1 \vee (\omega\omega^*)^\omega\omega_1^*, 3\right)^2.$$

Theorem (Todorcevic, 1983 and Malliaris, Shelah, 2013)

PID $+\mathfrak{t} > \aleph_1$ *implies* $\omega_1 \longrightarrow (\omega_1, \alpha)^2$ *for all ordinals* α.

General Problems (Raghavan, Todorcevic, 2014)

1. *Given a statement* φ *which is a consequence of* PID + $\mathrm{MA}_{\aleph_1}$ *, find a cardinal invariant* $\mathfrak{x}$ *such that* φ *is equivalent to* $\mathfrak{x} > \omega_1$ *over* ZFC + PID.
2. *Given a statement* φ *which is a consequence of* PID $+\mathfrak{p} > \omega_1$, *investigate whether* φ *is equivalent to* $\mathfrak{p} > \omega_1$ *over* ZFC + PID.

Question

1. *Does* $\mathfrak{b} = \aleph_1$ *imply* $\omega_1\omega \not\to (\omega_1\omega, 3)^2$*?*
2. *Does* $\text{⫯} = \aleph_1$ *imply* $\omega_1\omega \not\to (\omega_1\omega, 3)^2$*?*

Question (Larson)

1. *Does* $\omega_1^2 \not\to (\omega_1\omega, 4)^2$ *follow from a nontrivial cardinal characteristic assumption?*
2. *Is any cardinal characteristic assumption needed to prove* $\omega_1^2 \not\to (\omega_1^2, 3)^2$*?*

Question

Can the conclusion of the last Six-Alternatives-Theorem be strengthened(if necessary at the price of strengthening the assumptions)?

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