On maximal connected I-spaces

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Winter School in Abstract Analysis
Set Theory & Topology
Hejnice, Jan 28–Feb 4 2017
Maximal connected spaces

Definition

A topological space is called

- **maximal connected** [Thomas, 1968] if it is connected and has no connected strict expansion;
- **essentially connected** [Guthrie–Stone, 1973] if it is connected and every connected expansion has the same connected subsets.
Maximal connected spaces

Facts


- The real line is essentially connected [Hildebrand, 1967] and it has a maximal connected expansion [Simon, 1978], [Guthrie–Stone–Wage, 1978].

- No Hausdorff connected space with a dispersion point has a maximal connected expansion. [Guthrie–Stone, 1973]

- There are Hausdorff maximal connected spaces, but it is not known whether there are nondegenerate regular maximal connected spaces.
Implications between the classes

Definition

Recall the following properties of a topological space $X$.

- $X$ is *submaximal* if every its dense subset is open.
- $X$ is *nodec* if every its nowhere dense subset is closed.
- $X$ is *irresolvable* if it has no two disjoint dense subsets.
- $X$ is $T_{\frac{1}{2}}$ if every its singleton is open or closed.

We have the following implications.

- Maximal connected $\rightarrow$ Submaximal $\rightarrow$ Nodec
- Essentially connected $\rightarrow$ $T_{\frac{1}{2}}$ $\rightarrow$ Hereditarily irresolvable
A topological space $X$ is called *finitely generated* or *Alexandrov* if every intersection of open sets is open. Equivalently, if

$$\overline{A} = \bigcup_{x \in A} \{x\}$$

for every $A \subseteq X$. 
[Thomas, 1968] characterized finitely generated maximal connected spaces, we may reformulate the characterization as follows.

**Proposition**

Let $X$ be a finitely generated $T_{\frac{1}{2}}$ space. Let $I(X)$ be the set of all isolated points.

- The topology is uniquely determined by the bipartite graph $G_X$ with bipartition $\langle I(X), X \setminus I(X) \rangle$ and with an edge between $x \in I(X)$ and $y \in X \setminus I(X)$ if and only if $\{x\} \ni y$.
- $X$ is connected $\iff G_X$ is connected as a graph.
- $X$ is maximal connected $\iff G_X$ is a tree.

Therefore, finitely generated maximal connected spaces correspond to trees with fixed ordered bipartition.
Finitely generated maximal connected spaces

Figure: Examples of finitely generated maximal connected spaces.
Finitely generated maximal connected spaces

Figure: All nondegenerate maximal connected spaces with at most five elements.
I-spaces

Definition

Let $X$ be a topological space. By $I(X)$ we denote the set of all isolated points of $X$.

- $X$ is an *$I$-space* if $X \setminus I(X)$ is discrete.
- $X$ is *$I$-dense* if $I(X) = X$.
- $X$ is *$I$-flavored* if $I(X) \setminus I(X)$ is discrete.

I-spaces were considered in [Arhangel’skii–Collins, 1995].

We are interested in *maximal connected $I$-spaces*, a class containing finitely generated maximal connected spaces.
We have the following implications between the classes.

- The red part is a meet semilattice with respect to conjunction.
The green part collapses in the realm of maximal connected spaces.
Definition

Let \( \langle X_i : i \in I \rangle \) be an indexed family of topological spaces, \( \sim \) an equivalence on \( \sum_{i \in I} X_i \), and \( X := \sum_{i \in I} X_i / \sim \). We consider

- the canonical maps \( e_i : X_i \rightarrow X \),
- the canonical quotient map \( q : \sum_{i \in I} X_i \rightarrow X \),
- the set of gluing points \( S_X := \{ x \in X : |q^{-1}(x)| > 1 \} \),
- the gluing graph \( G_X \) with vertices \( I \sqcup S_X \) and edges of from \( s \rightarrow_x i \) where \( s \in S_X, i \in I \), and \( x \in X_i \) such that \( e_i(x) = s \).

We say that \( X \) is a tree sum if \( G_X \) is a tree, i.e. for every pair of distinct vertices there is a unique undirected path connecting them.

We just glue topological spaces in a way that the spaces are preserved, two spaces may be glued only at one point, and the global structure of connections forms a tree.
Proposition

A topological space $X$ is naturally homeomorphic to a tree sum of a family of its subspaces $\langle X_i : i \in I \rangle$ if and only if the following conditions hold.

1. $\bigcup_{i \in I} X_i = X$,
2. $X$ is inductively generated by embeddings $\{ e_i : X_i \to X \}_{i \in I}$,
3. $G$ is a tree, where $G$ is the graph on $S \sqcup I$ satisfying
   - $S := \{ x \in X : |\{ i \in I : x \in X_i \}| \geq 2 \}$,
   - $s \to i$ is an edge if and only if $s \in S$, $i \in I$, and $s \in X_i$. 
Definition

- We say that $A \subseteq X$ is an $I$-subset of $X$ if it is a union of an open discrete subset and a closed discrete subset of $X$.
- We say that (the gluing of) a tree sum is $I$-compatible if we never glue an isolated point to a non-isolated point.

Theorem [B.]

Let $X$ be a tree sum of nondegenerate spaces $\langle X_i : i \in I \rangle$. The following conditions are equivalent.

1. $X$ is maximal connected.
2. Every $X_i$ is maximal connected and $S_X$ is an $I$-subset of $X$.
3. Every $X_i$ is maximal connected, $S_X \cap X_i$ is an $I$-subset of $X_i$ for every $i \in I$, and the gluing is $I$-compatible.
4. Every $X_i$ is maximal connected and $X$ is essentially connected.
Proposition

Let $X$ be an $I$-compatible tree sum of spaces $\langle X_i : i \in I \rangle$. We have that $X$ is $P$ if and only if every $X_i$ is $P$ where $P$ is

- “finitely generated”,
- “an $I$-space”,
- “finitely generated maximal connected”,
- “a maximal connected $I$-space”.

Corollary

Besides the one-point space, finitely generated maximal connected spaces are exactly $I$-compatible tree sums of copies of the Sierpiński space.
There is a standard way of adding a closed discrete set.

**Definition**

Let $X$ be a topological space, $Y$ a set disjoint with $X$, and
$\mathcal{F} = \langle \mathcal{F}_y : y \in Y \rangle$ an indexed family of open filters on $X$. Let $\hat{X}$ be the space with universe $X \cup Y$ and the following topology:

$$A \subseteq \hat{X} \text{ is open } \iff \begin{cases} A \cap X \text{ is open in } X, \\ A \cap X \in \mathcal{F}_y \text{ for every } y \in A \cap Y. \end{cases}$$

- The space $\hat{X}$ is called the **OF-extension** of $X$ by $\mathcal{F}$.
- If every $\mathcal{F}_y$ is maximal, then $\hat{X}$ is called **MOF-extension**.
- If every $\mathcal{F}_y$ contains $I(X)$, then $\hat{X}$ is called **$I$-extension**.
- If both conditions hold, then $\hat{X}$ is called **ultrafilter $I$-extension**.
Remarks

- Let \( X \subseteq \hat{X} \) be topological spaces. \( \hat{X} \) is an OF-extension of \( X \) if and only if \( X \) is open dense and \( \hat{X} \setminus X \) is closed discrete nowhere dense in \( \hat{X} \).

- For I-extensions we may view the open filters \( \mathcal{F}_y \) containing \( I(X) \) as ordinary filters on \( I(X) \). Maximal open filters containing \( I(X) \) correspond to ultrafilters on \( I(X) \).

- I-spaces are precisely I-extensions of discrete spaces.

- OF-extensions preserve connectedness.
Proposition

Let $X$ be a maximal connected space. For every connected $A \subseteq X$ we have that $\overline{A}$ is a MOF-extension of $A$.

*Sketch of proof.*

- Both $A$ and $\overline{A}$ are maximal connected.
- $A$ is open dense in $\overline{A}$ and $\overline{A} \setminus A$ is closed discreet.
- $\overline{A}$ is an OF-extension of $A$.
- The extending filters have to be maximal.
OF-extensions and maximal connectedness

Observation

A topological space is open-hereditarily irresolvable if and only if \( \text{int}(A) \cup \text{int}(B) \) is dense for every its decomposition \( \langle A, B \rangle \).

Proposition

An OF-extension \( \langle \hat{X}, \tau \rangle \) of a maximal connected space \( X \) by a family of filters \( \langle F_y : y \in Y \rangle \) is maximal connected if and only if it is a MOF-extension of \( X \).

Sketch of proof of “\( \Leftarrow \)”:  

- Let \( A \subseteq X \) be non-open, \( \tau^* := \tau \cup \{A\} \).
- WLOG \( A \subseteq X \), and so \( \tau^* \upharpoonright X \) is disconnected.
- Let \( \langle U, V \rangle \) be a \( (\tau^* \upharpoonright X) \)-clopen decomposition of \( X \).
- \( \text{int}_\tau(U) \cup \text{int}_\tau(V) \) is \( \tau \)-dense.
- Every maximal filter \( F_y \) contains exactly one of \( U, V \).
- \( \langle \overline{U}^{\tau^*}, \overline{V}^{\tau^*} \rangle \) is a \( \tau^* \)-clopen decomposition of \( \hat{X} \).
Proposition

An OF-extension of a topological space $X$ is an $I$-space if and only if it an $I$-extension and $X$ is an $I$-space.

Corollary

Let $X$ be a maximal connected $I$-space.

- An OF-extension of $X$ is a maximal connected $I$-space if and only if it is an ultrafilter $I$-extension.
- $\bar{A}$ is an ultrafilter $I$-extension of $A$ for every connected $A \subseteq X$. 
We have described two constructions that preserve the property of being maximal connected I-space:

- I-compatible tree sums,
- ultrafilter I-extensions.

Therefore, we may build various maximal connected I-spaces inductively using the constructions.

Next we shall show how to deconstruct a maximal connected I-space in order to see whether it was inductively built using the constructions.
Intersections of connected subsets

We will need the following results.

**Theorem** [Neumann-Lara, Wilson; 1986]

Let $X$ be an **essentially connected** space. If $A, B \subseteq X$ are connected, then $A \cap B$ is connected as well.

**Corollary**

Let $X$ be an **maximal connected** space. If $A, B \subseteq X$ are disjoint and connected, then $|\overline{A} \cap \overline{B}| \leq 1$.

*Proof.*

We have $\overline{A} \cap \overline{B} \subseteq (\overline{A} \setminus A) \cup (\overline{B} \setminus B)$, which is a closed discrete set since $X$ is submaximal.
Towards characterization of maximal connected I-spaces

Definition

Let $X$ be a maximal connected space, let $\mathcal{D}$ be a decomposition of $X$ into connected subspaces.

- We define a graph $G_\mathcal{D}$ as follows: the vertices are the members of $\mathcal{D}$ and for $D \neq D' \in \mathcal{D}$ and $x \in X$, there is an edge $D \rightarrow_x D'$ if and only if $\overline{D} \cap D' \ni x$.
- We put $\mathcal{D}^+ := \{ \bigcup C : C$ is an undirected component of $G_\mathcal{D} \}$.

Proposition

Given the objects above, let $D \in \mathcal{D}^+$ and let $\mathcal{C}$ be the component of $G_\mathcal{D}$ such that $D = \bigcup \mathcal{C}$. We have that $G \upharpoonright \mathcal{C}$ is a tree and $D$ is the tree sum of its subspaces $\{ \overline{C} : C \in \mathcal{C} \}$ with closed discrete set of gluing points.
Towards characterization of maximal connected I-spaces

**Definition**

Let $X$ be a maximal connected space. We inductively define decompositions $D_\alpha$ and corresponding equivalences $E_\alpha$ for every $\alpha$.

- $D_0 := \{\{x\} : x \in X\}$,
- $D_{\alpha+1} := D_\alpha$
- $E_\alpha := \bigcup_{\beta<\alpha} E_\beta$ for limit $\alpha$.

We denote the smallest $\alpha$ such that $D_\alpha = D_{\alpha+1}$ by $\rho(X)$. 
Towards characterization of maximal connected $I$-spaces

**Theorem**

In the situation above suppose that $X$ is a maximal connected $I$-space. Let $D \in \mathcal{D}_\alpha$ for some $\alpha$.

1. $D$ is an $I$-compatible tree sum of ultrafilter $I$-extensions of some members of $\mathcal{D}_\beta$ if $\alpha = \beta + 1$. The ultrafilters are principal if $\beta = 0$, free otherwise.

2. $D$ is the direct limit of $\{ C \in \bigcup_{\beta < \alpha} \mathcal{D}_\alpha : C \subseteq D \}$ if $\alpha$ is limit.

Therefore, the members of $\mathcal{D}_{\rho(X)}$ are obtained by iteratively forming tree sums of ultrafilter $I$-extensions.

**Proposition**

Every maximal connected space having only finitely many nonisolated points is an $I$-space satisfying $|\mathcal{D}_1| < \omega$ and $|\mathcal{D}_2| \leq 1$. Therefore, it is a finite tree sum of free ultrafilter $I$-extensions of finitely generated maximal connected spaces.
Because of the previous results, a maximal connected I-space $X$ such that $|D_{\rho(X)}| \leq 1$ may be called *inductive*. We shall conclude with an example of a non-inductive maximal connected I-space.

**Example**

Let $f : X \rightarrow Y$ be a bijection between two disjoint sets, let $\mathcal{U}$ be a free ultrafilter on $X$. Let $\hat{X}$ be the I-extension of $X$ with discrete topology by the family $\langle \mathcal{F}_y : y \in Y \rangle$ where

$$\mathcal{F}_y := \{ U \in \mathcal{U} : f^{-1}(y) \in U \}$$

for every $y \in Y$.

The space $\hat{X}$ is an example of a non-inductive maximal connected I-space.


Thank you for your attention.

Highlights

- We are interested in *maximal connected spaces*.
- *Finitely generated* maximal connected spaces were characterized.
- Characterizing *maximal connected I-spaces* would generalize this.
- Constructions of *I-compatible tree sum* and *ultrafilter I-extension* preserve the property of being maximal connected I-space. We can build spaces inductively.
- Starting with points and considering how closures intersect, we obtain a *sequence of coarsening decompositions* into inductive connected subspaces.
- Not every maximal connected I-space is *inductive*.