Introduction to Haar-small sets

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joint work with Szymon Głąb, Eliza Jabłońska and Taras Banakh (in progress)

Background

$(G,+)$ - abelian Polish group

G is locally compact iff there exists regular invariant Borel measure (so-called Haar measure) which is unique up to multiplying by constant.

Hence in locally compact Polish groups we obtain natural notion of null sets.

If considered group is not locally compact, analoguos notion (so-called Haar null sets) were introduced by Christensen in 1972. His notion was rediscovered by Hunt, Sauer and Yorke in 1992 and since that time it was deeply investigated.

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Definitions

We say that set $A \subset G$ is *Haar-null*, or $A \in \mathcal{HN}(G)$, if there exists Borel hull $B \supset A$ and a Borel probalistic measure μ on G such that for any $g \in G$ we have $\mu(B+g)=0$. We say that μ witnesses the fact that A is Haar-null.

Theorem (Christensen)

Haar-null sets forms a proper σ-ideal. If G is locally compact, they coincide with Haar-measure null sets.

Observation

For any $A \in \mathcal{HN}$:

- measure which witnesses this fact is continuous,
- A has empty interior,
- there exist compactly (or even Cantorly) supported measure which witnesses this fact.

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Definitions

Darji, 2013

We say that set $A \subset G$ is *Haar-meager*, or $A \in \mathcal{HM}(G)$, if there exists Borel hull $B \supset A$ and a continuous function $h : 2^\omega \to G$ such that for any $g \in G$ we have $h^{-1}(B+g) \in \mathcal{M}_{2^\omega}$. We say that h witnesses the fact that A is Haar-meager.

Theorem (Darji)

Haar-meager sets forms a σ-subideal of meager sets. Those notions coincides iff G is locally compact.

We say that set $A \subset G$ is *Darji-Haar-null*, or $A \in \mathcal{DHN}(G)$, if there exists Borel hull $B \supset A$ and a continuous function $h : 2^\omega \to G$ such that for any $x \in G$ we have $h^{-1}(B+g) \in \mathcal{N}_{2^\omega}$. We say that h witnesses the fact that A is Darji-Haar-meager.

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Haar-small sets

Theorem

$\mathcal{DHN} = \mathcal{HN}$

Sketch of the proof.

Now it seems reasonable to consider following definition:

Definition

Let $\mathcal{I}$ be (σ-)ideal on 2^ω. We will say that $A \subset G$ is Haar-$\mathcal{I}$-small ($A \in \mathcal{HI}$) if there exists Borel? hull $B \supset A$ and continuous $f : 2^\omega \to G$ such that $f^{-1}(B + g) \in \mathcal{I}$ for all $g \in G$.

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Let ν be a continuous, fully supported Borel probabilistic measure on 2^ω. There exists order-preserving continuous function $f : 2^\omega \to 2^\omega$ for which $f^{-1}(\mathcal{N}_\nu) \subset \mathcal{N}_\lambda$.

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Corollary

If μ is σ-finite Borel measure on 2^ω, then $\mathcal{HN}_\mu = \mathcal{HN}$.

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Theorem Type 1

If $\mathcal{I}$ has property $P1$, then $\mathcal{HI}$ has property $P2$.

Conjecture

Assume that for any injection $j : \omega \to \omega$ and $A \in \mathcal{I}$ we have $\{x \in 2^\omega : x \circ j \in A\} \in \mathcal{I}$. Then $\mathcal{HI}$ is an ideal. If moreover $\mathcal{I}|_{<s>} \cong \mathcal{I}$ for each $s \in 2^{<\omega}$ we may obtain a σ-ideal.

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Theorem Type 2

If $\mathcal{I}$ has natural description (like $\mathcal{M}$, $\mathcal{N}$, porous or microscopic sets), then $\mathcal{HI}$ coincides at least in case of G being locally compact.

Example

Type 2 may fall. Set $G = (\mathbb{R}, +)$ and $\mathcal{I} := Fin_{2^\omega}$. Then $\mathcal{HI}$ contains Cantor sets of arbitrarly large Hausdorff dimension < 1.

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Thank you for your attention!