Universal sets for $\sigma$-ideals

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Let $X$ be a Polish space, $\omega^\omega$ denote the Baire space.

**Definition**

We say that a set $U \subseteq X \times \omega^\omega$ is universal for a family of sets $\mathcal{F} \subseteq P(X)$ if for every $F \in \mathcal{F}$ there exists $y \in \omega^\omega$ such that

$$U^y = \{x \in X : (x, y) \in U\} = F$$
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Widely known facts are that for each $\alpha < \omega_1$ there exists a universal $\Sigma^0_\alpha$ set for the family of $\Sigma^0_\alpha$ sets and that there exists an analytic universal set for a family of analytic sets.
Let $\mathcal{I} \subseteq P(X)$ be a nontrivial $\sigma$-ideal possessing a Borel base.

**Definition**

We say that a set $U \subseteq X \times \omega^\omega$ is universal for the $\sigma$-ideal $\mathcal{I}$ if a family of horizontal slices $\{U^y : y \in \omega^\omega\}$ is a Borel base of $\mathcal{I}$.

We are interested in finding universal sets of possibly low complexity.
Theorem

There are Borel universal sets of minimal complexity for

- $\mathcal{M}$ - a family of meager sets;
- $\mathcal{N}$ - a family of null subsets of $2^\omega$;
- $\mathcal{E}$ - a $\sigma$-ideal generated by closed null subsets of $2^\omega$;
- $\sigma$-ideal of countable sets.
Let $X$ be a Polish space. We will start with constructing a universal open set $U \subseteq X \times \omega$ for open and dense subsets of $X$.

$\{B_n \mid n \in \omega\}$ - enumeration of basic open sets.

Let us define $K : \omega \times \omega \to \omega$ in the following way:

$K(n,0) = \min\{k \mid B_k \subseteq B_n\}$,

$K(n,m+1) = \min\{k \mid B_k \subseteq B_n \land k > K(n,m)\}$.

$K(n,m)$ gives a number of the $(m+1)$-st basic open set contained in $B_n$ with respect to our enumeration.

Let us set:

$(x,y) \in U \iff x \in \bigcup_{n \in \omega} B_{K(n,y(n))}$.
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**$F_\sigma$ universal set for the category**

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- Let us set:

$$(x, y) \in U \iff x \in \bigcup_{n \in \omega} B_{K(n, y(n))}.$$
Now let us fix a bijection $\omega \times \omega$ and set a homeomorphism $h : \omega^\omega \to \omega^{\omega^\omega}$ given by a formula:

$$(h(x)(m))(n) = x(b(m, n)),$$

for all $x \in \omega^\omega$. 

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$$(h(x)(m))(n) = x(b(m, n)),$$

for all $x \in \omega^\omega$.

Finally let us define a set $G$:

$$(x, y) \in G \iff x \in \bigcap_{n \in \omega} U^{h(y)}(n)$$

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- Let $\{B_n : n \in \omega\}$ by an enumeration of basic clopen sets.
- Let $2^\omega \times \omega^\omega \supseteq U = \{(x, y) : x \in \bigcup_{n \in \omega} B_{y(n)}\}$ be a universal open set with respect to our enumeration.
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- Let us fix $\epsilon > 0$ and consider a set $V = \{y \in \omega^\omega : \lambda(U^y) \leq \epsilon\}$.
- $V$ is closed so there is a continuous function $f : \omega^\omega \to V$. Let us set:

$$U_{\epsilon} = (Id \times f)^{-1}[2^\omega \times V],$$

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- Finally let us define:

$$(x, y) \in G \iff x \in \bigcap_{n \in \omega} U_{\frac{1}{n+1}}^{h(y)(n)}.$$

$G$ is the set.
Theorem

Let us assume that the base of $\mathcal{I}$ is contained in the class $\Sigma^0_\alpha$ and let $U$ be universal $\Sigma^0_\alpha$ set for $\Sigma^0_\alpha$ sets. Then if a set $\{y \in \omega^\omega : B^y \in \mathcal{I}\}$ is analytic, then there is a universal $\Sigma^0_\alpha$ set for $\mathcal{I}$. The same holds for the class $\Pi^0_\alpha$.
Thank you for your attention!