Universal sets for σ-ideals

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Let X be a Polish space, ω^ω denote the Baire space.

Definition

We say that a set $U \subseteq X \times \omega^\omega$ is universal for a family of sets $\mathcal{F} \subseteq P(X)$ if for every $F \in \mathcal{F}$ there exists $y \in \omega^\omega$ such that

$$U^y = \{x \in X : (x, y) \in U\} = F$$

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Widely known facts are that for each $\alpha < \omega_1$ there exists a universal Σ^0_α set for the family of Σ^0_α sets and that there exists an analytic universal set for a family of analytic sets.

Let $\mathcal{I} \subseteq P(X)$ be a nontrivial σ-ideal possesing a Borel base.

Definition

We say that a set $U \subseteq X \times \omega^\omega$ is universal for the σ-ideal $\mathcal{I}$ if a family of horizontal slices $\{U^y : y \in \omega^\omega\}$ is a Borel base of $\mathcal{I}$.

We are interested in finding universal sets of possibly low complexity.

Results

Theorem

There are Borel universal sets of minimal complexity for

- $\mathcal{M}$- *a family of meager sets;*
- $\mathcal{N}$- *a family of null subsets of* 2^ω*;*
- $\mathcal{E}$- *a* σ*-ideal generated by closed null subsets of* 2^ω*;*
- σ*-ideal of countable sets.*

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- Let us define $K : \omega \times \omega \to \omega$ in the following way:

$$K(n,0) = min\{k : B_k \subseteq B_n\},$$
$$K(n,m+1) = min\{k : B_k \subseteq B_n \ \wedge \ k > K(n,m)\}.$$

$K(n,m)$ gives a number of the $(m+1)st$ basic open set contained in B_n with respect to our enumeration.

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  $K(n,m)$ gives a number of the $(m+1)st$ basic open set contained in B_n with respect to our enumeration.
- Let us set:

$$(x,y) \in U \Leftrightarrow x \in \bigcup_{n \in \omega} B_{K(n,y(n))}.$$

F_σ universal set for the category continued

- Now let us fix Let b be a bijection $\omega \times \omega$ and ω and set a homeomorphism $h : \omega^\omega \to \omega^{\omega^\omega}$ given by a formula:

$$(h(x)(m))(n) = x(b(m, n)),$$

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- Finally let us define a set G:

$$(x,y) \in G \Leftrightarrow x \in \bigcap_{n \in \omega} U^{h(y)(n)}$$

G is a G_δ universal set for dense G_δ subsets of X, so G^c is the desired set.

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- Let $2^\omega \times \omega^\omega \supseteq U = \{(x, y) : x \in \bigcup_{n\in\omega} B_{y(n)}\}$ be a universal open set with respect to our enumeration.

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- V is closed so there is a continuous function $f : \omega^\omega \to V$. Let us set:

$$U_\epsilon = (Id \times f)^{-1}[2^\omega \times V],$$

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- Finally let us define:

$$(x, y) \in G \Leftrightarrow x \in \bigcap_{n\in\omega} U^{h(y)(n)}_{\frac{1}{n+1}}.$$

G is the set.

Theorem

Let us assume that the base of $\mathcal{I}$ is contained in the class Σ^0_α and let U be universal Σ^0_α set for Σ^0_α sets. Then if a set $\{y \in \omega^\omega : B^y \in \mathcal{I}\}$ is analytic, then there is a universal Σ^0_α set for $\mathcal{I}$. The same holds for the class Π^0_α.

Thank you for your attention!