Extending Baire-one functions on compact spaces

Olena Karlova

Chernivtsi National University
Definitions and notations

Let $B_1(X)$ be the collection of all Baire-one functions $f : X \rightarrow \mathbb{R}$ on a topological space $X$. 
Definitions and notations

Let $B_1(X)$ be the collection of all Baire-one functions $f : X \to \mathbb{R}$ on a topological space $X$.

A subspace $E$ of a topological space $X$ is called

- $B_1$-embedded in $X$, if any function $f \in B_1(E)$ can be extended to a Baire-one function $g : X \to \mathbb{R}$;
Definitions and notations

Let $B_1(X)$ be the collection of all Baire-one functions $f : X \to \mathbb{R}$ on a topological space $X$.

A subspace $E$ of a topological space $X$ is called

- $B_1$-embedded in $X$, if any function $f \in B_1(E)$ can be extended to a Baire-one function $g : X \to \mathbb{R}$;
- $B_1^*$-embedded in $X$, if any bounded function $f \in B_1(E)$ can be extended to a Baire-one function $g : X \to \mathbb{R}$. 


Theorem (O. Kalenda and J. Spurný, 2005)

Let $E$ be a Lindelöf hereditarily Baire subset of a completely regular space $X$ and $f : E \to \mathbb{R}$ be a Baire-one function. Then there exists a Baire-one function $g : X \to \mathbb{R}$ such that $g = f$ on $E$. 

If $A$ and $B$ are disjoint dense subsets of $E = \mathbb{Q} \setminus [0, 1]$ such that $E = A \cup B$ and $X = [0, 1]$ or $X = \beta E$, then the characteristic function $f = \chi_A : E \to \mathbb{R}$ can not be extended to a Baire-one function on $X$. 


History

Theorem (O. Kalenda and J. Spurný, 2005)

Let $E$ be a Lindelöf hereditarily Baire subset of a completely regular space $X$ and $f : E \to \mathbb{R}$ be a Baire-one function. Then there exists a Baire-one function $g : X \to \mathbb{R}$ such that $g = f$ on $E$.

If $A$ and $B$ are disjoint dense subsets of $E = \mathbb{Q} \cap [0, 1]$ such that $E = A \cup B$ and $X = [0, 1]$ or $X = \beta E$, then the characteristic function $f = \chi_A : E \to \mathbb{R}$ can not be extended to a Baire-one function on $X$. 
A question

Question (O. Kalenda and J. Spurný, 2005)
Is any hereditarily Baire completely regular space $X$ $B_1$-embedded in $\beta X$?
Let $X$ be a topological space and $(Y, d)$ be a metric space. A map $f : X \to Y$ is called
$\varepsilon$-fragmented for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\text{diam}f(U) < \varepsilon$;
Let $X$ be a topological space and $(Y, d)$ be a metric space. A map $f : X \to Y$ is called

- **$\varepsilon$-fragmented** for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\text{diam}f(U) < \varepsilon$;

- **fragmented** if $f$ is $\varepsilon$-fragmented for all $\varepsilon > 0$. 

Functionally fragmented maps

**Proposition**

Let $X$ be a topological space, $(Y, d)$ be a metric space and $\varepsilon > 0$. For a map $f : X \rightarrow Y$ the following conditions are equivalent:

1. $f$ is $\varepsilon$-fragmented;
2. there exists a sequence $\mathcal{U} = (U_\xi : \xi \in [0, \alpha))$ in $X$ of open sets such that
   - $\text{diam}_f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha)$;
   - $\emptyset = U_0 \subset U_1 \subset U_2 \subset \ldots$;
   - $U_\gamma = \bigcup_{\xi < \gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha)$. 


Functionally fragmented maps

An $\varepsilon$-fragmented map $f : X \to Y$ is

\begin{itemize}
  \item \textit{functionally $\varepsilon$-fragmented} if $\mathcal{U}$ can be chosen such that every $U_\xi$ is functionally open in $X$;
\end{itemize}
Functionally fragmented maps

An $\varepsilon$-fragmented map $f : X \to Y$ is

- *functionally $\varepsilon$-fragmented* if $U$ can be chosen such that every $U_\xi$ is functionally open in $X$;
- *functionally $\varepsilon$-countably fragmented* if $U$ can be chosen to be countable;
Functionally fragmented maps

An \( \varepsilon \)-fragmented map \( f : X \to Y \) is

- **functionally \( \varepsilon \)-fragmented** if \( \mathcal{U} \) can be chosen such that every \( U_\xi \) is functionally open in \( X \);

- **functionally \( \varepsilon \)-countably fragmented** if \( \mathcal{U} \) can be chosen to be countable;

- **functionally countably fragmented** if \( f \) is functionally \( \varepsilon \)-countably fragmented for all \( \varepsilon > 0 \).
Functionally fragmented maps

- functional countable fragmentability
- functional fragmentability
- countable fragmentability
- fragmentability
- continuity
- Baire-one
Functionally countably fragmented maps

Proposition

Let $X$ be a topological space, $(Y, d)$ be a metric space, $\varepsilon > 0$ and $f : X \to Y$ be a map. If

- $Y$ is separable and $f$ is continuous, or
- $X$ is hereditarily Lindelöf and $f$ is fragmented, or
- $X$ is compact and $f \in B_1(X, Y)$,

then $f$ is functionally countably fragmented.
Theorem (K., 2016)

Let $X$ be a completely regular space. For a Baire-one function $f : X \to \mathbb{R}$ the following conditions are equivalent:

- $f$ is functionally countably fragmented;
- $f$ can be extended to a Baire-one function on $\beta X$. 

Theorem (K. and V. Mykhaylyuk, 2016)

There exists a completely regular scattered (and hence hereditarily Baire) space $X$ and a Baire-one function $f : X \to [0,1]$ which cannot be extended to a Baire-one function on $\beta X$. 

Extension properties of fragmented maps
Theorem (K., 2016)

Let $X$ be a completely regular space. For a Baire-one function $f : X \to \mathbb{R}$ the following conditions are equivalent:

- $f$ is functionally countably fragmented;
- $f$ can be extended to a Baire-one function on $\beta X$.

Theorem (K. and V. Mykhaylyuk, 2016)

There exists a completely regular scattered (and hence hereditarily Baire) space $X$ and a Baire-one function $f : X \to [0, 1]$ which can not be extended to a Baire-one function on $\beta X$. 
Applications of extension theorem. $B_1$-embedding vs. $B_1^*$-embedding

**Theorem (K., 2013)**

Let $X$ be a hereditarily Baire space and $E$ be a perfectly normal Lindelöf subspace with a hereditary countable π-base. Then the following conditions are equivalent:

1. $E$ is $B_1^*$-embedded in $X$;
2. $E$ is $B_1$-embedded in $X$. 
Applications of extension theorem. $B_1$-embedding vs. $B^*_1$-embedding

**Corollary 1.**

For a countable subspace $E$ of a metrizable space $X$ the following conditions are equivalent:

1. $E$ is $B^*_1$-embedded in $X$;
2. $E$ is $G_\delta$ in $X$. 
Applications of extension theorem. $B_1$-embedding vs. $B_1^*$-embedding

**Corollary 1.**
For a countable subspace $E$ of a metrizable space $X$ the following conditions are equivalent:
1. $E$ is $B_1^*$-embedded in $X$;
2. $E$ is $G_δ$ in $X$.

**Corollary 2.**
Any countable hereditarily irresolvable completely regular space $X$ is $B_1^*$-embedded in $βX$ and is not $B_1$-embedded in $βX$. 
Applications of extension theorem. Baire classification

Corollary 3.

Every functionally countably fragmented function $f : X \rightarrow \mathbb{R}$ defined on a topological space $X$ is Baire-one.
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]

- Functional countable fragmentability
- Functional fragmentability
- Fragmentability
- Countable fragmentability
- Baire-one
- Continuity
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]

- functional countable fragmentability
- functional fragmentability
- fragmentability
- countable fragmentability
- Baire-one
- continuity
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]

- Functional countable fragmentability
- Functional fragmentability
- Fragmentability
- Countable fragmentability
- Baire-one
- Continuity

\[ X \text{ is hereditarily Baire} \]
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]
Applications of extension theorem. Baire classification

\[ Y = \mathbb{R} \]