Extending Baire-one functions on compact spaces

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Definitions and notations

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- B_1-embedded in X, if any function $f \in \mathrm{B}_1(E)$ can be extended to a Baire-one function $g : X \to \mathbb{R}$;
- B_1^*-embedded in X, if any bounded function $f \in \mathrm{B}_1(E)$ can be extended to a Baire-one function $g : X \to \mathbb{R}$.

History

Theorem (O. Kalenda and J. Spurný, 2005)

Let E be a Lindelöf hereditarily Baire subset of a completely regular space X and $f : E \to \mathbb{R}$ be a Baire-one function. Then there exists a Baire-one function $g : X \to \mathbb{R}$ such that $g = f$ on E.

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◆ if A and B are disjoint dense subsets of $E = \mathbb{Q} \cap [0, 1]$ such that $E = A \cup B$ and $X = [0, 1]$ or $X = \beta E$, then the characteristic function $f = \chi_A : E \to \mathbb{R}$ can not be extended to a Baire-one function on X.

A question

Question (O. Kalenda and J. Spurný, 2005)

Is any hereditarily Baire completely regular space X B_1-embedded in βX?

Functionally fragmented maps

Let X be a topological space and (Y, d) be a metric space.
A map $f : X \to Y$ is called

- ε-*fragmented* for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\operatorname{diam} f(U) < \varepsilon$;

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- *fragmented* if f is ε-fragmented for all $\varepsilon > 0$.

Functionally fragmented maps

Proposition

Let X be a topological space, (Y, d) be a metric space and $\varepsilon > 0$. For a map $f : X \to Y$ the following conditions are equivalent:

1. f is ε-fragmented;
2. there exists a sequence $\mathscr{U} = (U_\xi : \xi \in [0, \alpha))$ in X of open sets such that
   - $\operatorname{diam} f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha)$;
   - $\emptyset = U_0 \subset U_1 \subset U_2 \subset \ldots;$
   - $U_\gamma = \bigcup_{\xi<\gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha)$.

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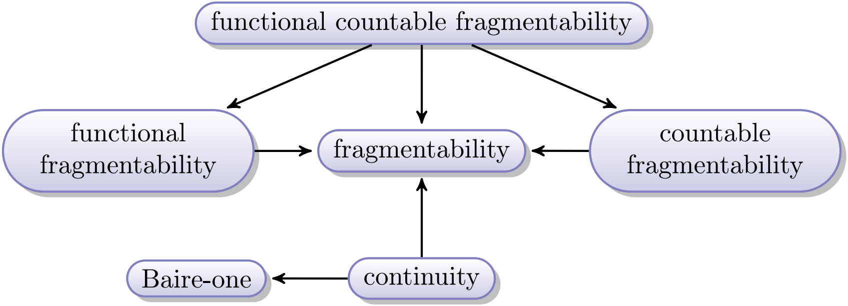
- *functionally ε-fragmented* if $\mathscr{U}$ can be chosen such that every U_ξ is functionally open in X;
- *functionally ε-countably fragmented* if $\mathscr{U}$ can be chosen to be countable;
- *functionally countably fragmented* if f is functionally ε-countably fragmented for all $\varepsilon > 0$.

Functionally fragmented maps



Functionally countably fragmented maps

Proposition

Let X be a topological space, (Y, d) be a metric space, $\varepsilon > 0$ and $f : X \to Y$ be a map. If

- Y is separable and f is continuous, or
- X is hereditarily Lindelöf and f is fragmented, or
- X is compact and $f \in \mathrm{B}_1(X, Y)$,

then f is functionally countably fragmented.

Extension properties of fragmented maps

Theorem (K., 2016)

Let X be a completely regular space. For a Baire-one function $f : X \to \mathbb{R}$ the following conditions are equivalent:

- f is functionally countably fragmented;
- f can be extended to a Baire-one function on βX.

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- f is functionally countably fragmented;
- f can be extended to a Baire-one function on βX.

Theorem (K. and V.Mykhaylyuk, 2016)

There exists a completely regular scattered (and hence hereditarily Baire) space X and a Baire-one function $f : X \to [0, 1]$ which can not be extended to a Baire-one function on βX.

Applications of extension theorem. B_1-embedding vs B_1^*-embedding

Theorem (K., 2013)

Let X be a hereditarily Baire space and E be a perfectly normal Lindelöf subspace with a hereditary countable π-base. Then the following conditions are equivalent:

1. E is B_1^*-embedded in X;
2. E is B_1-embedded in X.

Applications of extension theorem. B_1-embedding vs B_1^*-embedding

Corollary 1.

For a countable subspace E of a metrizable space X the following conditions are equivalent:

1. E is B_1^*-embedded in X;
2. E is G_δ in X.

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Corollary 1.

For a countable subspace E of a metrizable space X the following conditions are equivalent:

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Corollary 2.

Any countable hereditarily irresolvable completely regular space X is B_1^*-embedded in βX and is not B_1-embedded in βX.

Applications of extension theorem. Baire classification

Corollary 3.

Every functionally countably fragmented function $f : X \to \mathbb{R}$ defined on a topological space X is Baire-one.

Applications of extension theorem. Baire classification

$$Y = \mathbb{R}$$

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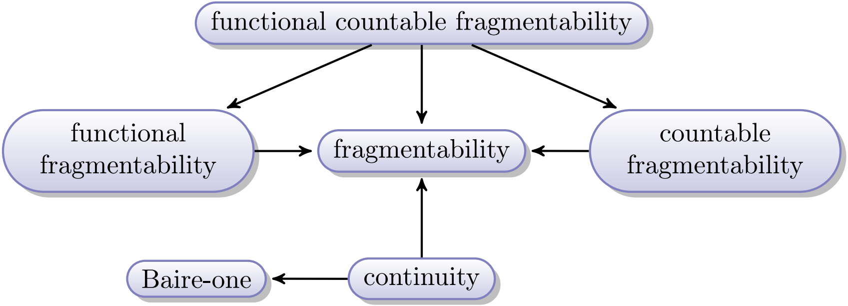

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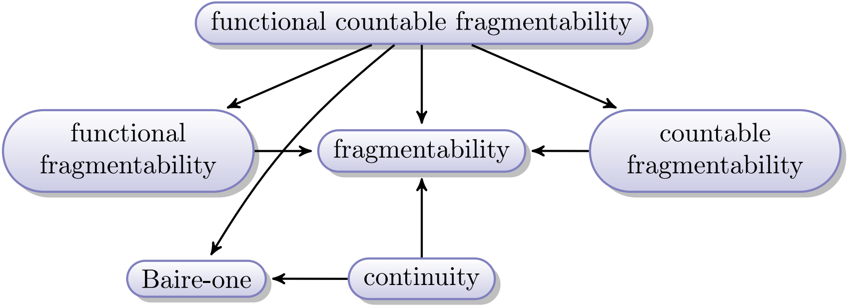

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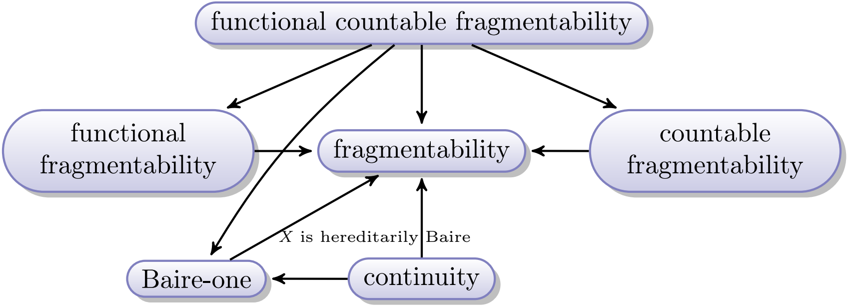

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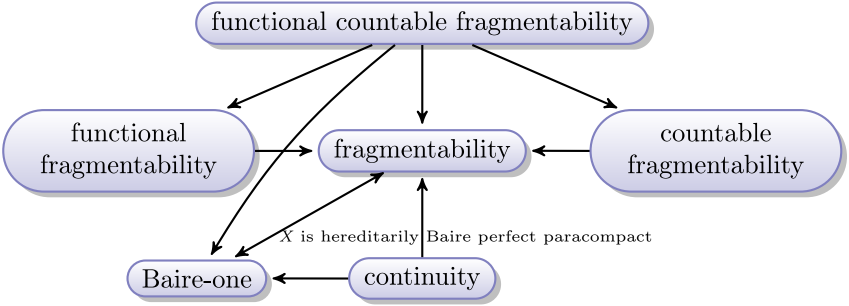

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