Extension of Baire-one functions

Let $\mathrm{B}_1(X)$ be the collection of all Baire-one functions on a topological space X.

A subspace E of a topological space X is called B_1-*embedded* (B_1^*-*embedded*) in X, if any (bounded) function $f \in \mathrm{B}_1(E)$ can be extended to $g \in \mathrm{B}_1(X)$; 1-*embedded in* X, if any functionally G_δ-set in E can be extended to a functionally G_δ-set in X; *ambiguously* 1-*embedded in* X, if any functionally ambiguous set in E can be extended to a functionally ambiguous set in X; *well* 1-*embedded in* X, if for any functionally G_δ-set $A \subseteq X$ disjoint with E there exists a function $f \in \mathrm{B}_1(X)$ such that $E \subseteq f^{-1}(0)$ and $A \subseteq f^{-1}(1)$.

We show that a subspace E of a topological space X is B_1^*-embedded in X if and only if E is ambiguously 1-embedded in X. We prove that E is B_1-embedded in X if and only if E is 1-embedded and well 1-embedded in X. Moreover, any countable hereditarily irresolvable completely regular space is B_1^*-embedded in βX and is not B_1-embedded in βX.

Recall that a function $f : X \to \mathbb{R}$ is *fragmented* if for every $\varepsilon > 0$ and for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\mathrm{diam} f(U) < \varepsilon$. Notice that every Baire-one real-valued function defined on a hereditarily Baire space is fragmented. We prove that any fragmented function defined on a countable completely regular space X can be extended to a Baire-one function defined on βX.