Brouwer and Cardinalities

Tá scéilín agam

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L. E. J. Brouwer (27 February 1881 – 2 December 1966)

Last year was a Brouwer year in The Netherlands; 50 years after his death.

This talk is related to an article for a special issue of *Indagationes Mathematicae*, dedicated to Brouwer.

Die moeglichen Maechtigkeiten

I will discuss a paper from 1908, based on a short section in Brouwer's doctoral thesis *Over de Grondslagen der Wiskunde*.

That paper ends with

Von anderen unendlichen Mächtigkeiten, ab der abzählbaren, der abzählbar-unfertigen, und der kontinuierlichen, kann gar keine Rede sein.

Die moeglichen Maechtigkeiten

I will discuss a paper from 1908, based on a short section in Brouwer's doctoral thesis *Over de Grondslagen der Wiskunde*.

That paper ends with

Of other infinite powers than the countable, the countable unfinished, and the continuous, we cannot speak.

What the ... does that mean?

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Ur-intuition (primordial intuition)

The paper opens with a paragraph with a description of the 'primordial mathematical intuition'.

"One can think of a Second, not on its own, but while remembering the First; the First and the Second are kept together."

It makes no sense at all, to me anyway.

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ω and η

A bit more concrete: that Intuition has in itself the possibility to develop

the order type ω:
"when one thinks of the whole Intuition as a new first then one can think of a new Second, that one calls 'Three', and so on"

ω and η

A bit more concrete: that Intuition has in itself the possibility to develop

the order type η:
"when one thinks of the Intuition as a transit between the First and the Second, the 'insertion' has come about"

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Some comments

These steps look quite finite, but Brouwer seems to allow for infinitely many instances, and even to consider those completed.

It looks like he accepted the actual infinity.

What more?

"Everything constructed can be seen as a new unit and this explains the richness of the infinite fullness of the mathematically possible systems"

Everything can be traced back to ω and η.

Brouwer claims that one can only build *countable* discrete systems in this way.

But . . .

. . . there is room for the uncountable

One can describe a method that creates from every given countable set in the system a new element of the system.

Two claims:

- one can only construct countable sets in this way, *as everywhere in Mathematics*
- one cannot build the full system in this way because it cannot be countable

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. . . there is room for the uncountable

It is incorrect to call the whole system a mathematical set for it cannot be built finished.

Examples: the whole of the numbers of the second number class, the whole of the definable points on the continuum, the whole of the mathematical systems.

The Continuum

One can consider the continuum as a matrix of points or units and assume that two points can be considered distinct if and only if their positions can be distinguished on a certain scale of order type η.

The Continuum

Continuum: the real line?

Matrix: the place in which anything is developed or formed

Scale: some subset of order type η (points are distinct if there is a point from the scale between them)

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The Continuum

One then observes that the thus defined continuum will never let itself be exhausted as a matrix of points, and one has to add the possibility of overlaying a scale of order type η with a continuum to the method for building mathematical systems.

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The Continuum

Why can the continuum never be exhausted?

How is the continuum defined?

We get our first 'concrete' construction step: complete a set of order type η to a continuum.

Subsets of the Continuum

Brouwer next 'proves' that a subset of the continuum is either countable or of the same power of the continuum.

Really? How?

Subsets of the Continuum

Let M be a subset of the continuum; we assume $M \subseteq [0, 1]$.

In M there is a countable subset M_1, with whose help M is defined.

M_1 determines a subtree of the binary tree, say using intervals of the form $[a2^{-n}, (a+1)2^{-n}]$.

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Subsets of the Continuum

A Cantor-Bendixson analysis of the tree leads to two cases.

1. At the end the tree is empty; then M is countable
2. At the end the tree is nonempty; then M_1 contains subsets of type η.
   Then M arises from such sets by overlaying a continuum; and possibly deleting countably many points.
   So, in this case M has the power of the continuum.

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Conclusion of the paper

Thus there exist just one power for mathematical infinite sets, to wit the countable. One can add to these:

1. *the countable-unfinished*, but by this is meant a *method*, not a set
2. *the continuous*, by this is certainly meant something finished, but only as *matrix*, not as a set.

Of other infinite powers, besides the countable, the countable-unfinished, and the continuous, one cannot speak.

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Zentralblatt 40.0099.04

Referent bekennt, daß die Betrachtungen des Verfassers ihm nicht völlig klar sind; er beschränkt sich also darauf, seine Schlüsse wörtlich abzuschreiben:

Zentralblatt 40.0099.04

"Es existiert nur eine Mächtigkeit für mathematische unendliche Mengen, nämlich die abzählbare. Man kann aber hinzufügen:
1. die abzählbar-unfertige, aber dann wird eine Methode, keine Menge gemeint;
2. die kontinuierliche, dann wird freilich etwas Fertiges gemeint, aber nur als Matrix, nicht als Menge.
Von anderen unendlichen Mächtigkeiten, ab der abzählbaren, der abzählbar-unfertigen, und der kontinuierlichen, kann gar keine Rede sein." Reviewer: Vivanti, Prof. (Pavia)

(*Jahrbuch über die Fortschritte der Mathematik*)

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The thesis

The thesis gives a few more details.
For example: how to ‘overlay’ a continuum over a nowhere dense subset of order type η.
In short: with holes; a bit like the Cantor set without the left-hand end points.

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The thesis

And the conclusion: "it appears that this solves the continuum problem, by adhering strictly to the insight: 'one cannot speak of the continuum as a point set other than in relation to a scale of order type η' "

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Standard reading

It looks like Brouwer allows himelf to construct

- countable sets
- closed sets
- unions, intersections, differences

So we can read the paper as a proof that closed sets in $\mathbb{R}$ are countable or of cardinality continuum.

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Context

This was Brouwer's first attempt to articulate what he thought mathematics was or should be.

It would take decades to 'get it right'.

Don't forget that it lead to many notions that have become important in (theoretical) computer science for example.

Caveat

J. Korevaar related an anedote about Brouwer.

At one of the monthly meetings of the Dutch Mathematical Society in the 1950s some mathematicians were explaining to each other what Brouwer had meant in some paper.
Unnoticed by everyone Brouwer had entered the room and after a while he ran to the board exclaiming:

Caveat

"You have all misunderstood me!"

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Light reading

Website: fa.its.tudelft.nl/~hart

L. E. J. Brouwer,
Die moeglichen Maechtigkeiten, Atti del IV congresso internazionale dei matematici, (1908) 569–571.

Klaas Pieter Hart,
Brouwer and Cardinalities,
https://arxiv.org/abs/1612.06606.