Iterated forcing with side conditions

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Properness

(Shelah) A forcing notion $\mathcal{P}$ is *proper* iff for every cardinal $\theta > |\mathcal{P}|$, every countable $N \prec H(\theta)$ such that $\mathcal{P} \in N$ and every $p \in N \cap \mathcal{P}$ there is some $q \leq p$ such that

$$q \Vdash_{\mathcal{P}} D \cap \dot{G} \cap N \neq \emptyset$$

for every dense $D \subseteq \mathcal{P}$ such that $D \in N$.

We say that q is $(N, \mathcal{P})$–*generic*.

Note: $\mathcal{P}$ is proper iff the above holds for some $\theta > |\mathcal{P}|$.

Proper forcing is nice:

- Proper forcing notions preserve ω_1.
- Properness is preserved under countable support (CS) iterations.

Hence, granted the existence of a supercompact cardinal, one can build a model of PFA, the forcing axiom for proper forcings relative to collection of $\aleph_1$–many dense series (Baumgartner).

PFA: For every proper $\mathcal{P}$ and for every collection $\{D_i : i < \omega_1\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all i.

PFA has many consequences. One of them is $2^{\aleph_0} = \aleph_2$.

Problem: Force some consequence of PFA or, for that matter, something we can force by iterating non–c.c.c. proper forcing, together with $2^{\aleph_0} > \aleph_2$.

Countable support iterations won't do. In fact, at stages of uncountable cofinality we are adding generics, over all previous models, for $\mathrm{Add}(1, \omega_1)$ (= adding a Cohen subset of ω_1); in particular we are collapsing the continuum of all those previous models to $\aleph_1$. Hence, in the final model necessarily $2^{\aleph_0} \leq \aleph_2$.

Bigger support won't work either: The preservation lemma for properness doesn't work in the present context.

Finite support iterations won't work either; in fact, any finite support ω–length iteration of non–c.c.c. forcings collapses ω_1.

Side conditions

Rough idea: We're interested in forcing with a non-proper $\mathcal{P}$, and we would really like it to be proper. We can look at some similar forcing $\mathcal{P}^*$ *which incorporates countable models* as side conditions and is thereby proper.

First example perhaps Baumgartner's forcing for adding a club of ω_1 with finite condition.

Method made explicit in work of Todorčević from the 1980's.

Typical examples: Conditions in $\mathcal{P}^*$ are pairs of the form $(w, \mathcal{N})$, where

- w is the *working part* (adding the object we are ultimately interested in).
- $\mathcal{N}$ is a finite $\in$–chain (i.e., can be ordered as $(N_i)_{i<n}$ with $N_i \in N_{i+1}$ for all i) of elementary submodels of some suitable $H(\chi)$ containing all relevant objects.
- w is "generic for all members of $\mathcal{N}$".

Extension: $(w_1, \mathcal{N}_1) \leq (w_0, \mathcal{N}_0)$ iff

- w_1 extends w_0 (in some natural way), and
- $\mathcal{N}_0 \subseteq \mathcal{N}_1$.

Typical proof of properness:

- Start with $(w, \mathcal{N}) \in N$, N countable, $N \prec H(\theta)$ for large enough θ.
- **Add** $N \cap H(\chi)$ **to** $(w, \mathcal{N})$. That is, build $(\bar{w}, \mathcal{N} \cup \{N \cap H(\chi)\})$, where $\bar{w}$ is perhaps some extension of w.
- Prove that $(\bar{w}, \mathcal{N} \cup \{N \cap H(\chi)\})$ is $(N, \mathcal{P}^*)$–generic.

Example: Measuring one club–sequence by finite conditions.

Weak Club Guessing at ω_1 (WCG):
There is a ladder system $(C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ (i.e., for all δ, $C_\delta \subseteq \delta$ is cofinal in δ and of order type ω) such that for every club $D \subseteq \omega_1$ there is some δ such that $|D \cap C_\delta| = \aleph_0$.

WCG is a very weak version of Jensen's $\diamondsuit$.

Killing one instance of WCG:

Let $\vec{C} = (C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ ladder system. Let $\mathcal{P}_{\vec{C}}$ be as follows: Conditions are pairs (f, b) such that

(1) $f \subseteq \omega_1 \times \mathrm{Lim}(\omega_1)$ is a finite function that can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \mathrm{Lim}(\omega_1)$.

(2) $\mathrm{dom}(b) = \mathrm{dom}(f)$ and $b(\xi) < f(\xi)$ for each $\xi \in \mathrm{dom}(b)$.

(3) For each $\xi \in \mathrm{dom}(b)$, $C_{f(\xi)} \cap \mathrm{range}(f \restriction \xi) \subseteq b(\xi)$.

Extension: $(f_1, b_1) \leq (f_0, b_0)$ iff

- $f_0 \subseteq f_1$ and
- $b_0 \subseteq b_1$.

(This is the natural version of Baumgartner's forcing for adding a club with finite conditions incorporating promises to avoid relevant C_δ's.)

$\mathcal{P}_{\vec{C}}$ is proper:

Let $(f, b) \in N$, where $N \prec H(\theta)$ for quite large θ.
Let $\delta_N = N \cap \omega_1 \in \omega_1$. Then $(f \cup \{(\delta_N, \delta_N)\}, b)$ is $(N, \mathcal{P}_{\vec{C}})$–generic:

Let (f', b') extend $(f \cup \{(\delta_N, \delta_N)\}, b)$ and let $D \subseteq \mathcal{P}$ dense and open, $D \in N$. By extending (f', b') if necessary we may assume $(f', b') \in D$.

Note: $f' \restriction \delta_N$, $b' \restriction \delta_N \in N$. In N pick θ_0 large enough and let $(M_\nu)_{\nu<\omega_1}$ $\subseteq$–continuous chain of countable elementary substructures of $H(\theta_0)$ containing $f' \restriction \delta_N$, $b' \restriction \delta_N$ and D.

$(\delta_{M_\nu})_{\nu<\delta_N}$ is a club of δ_N of order type δ_N. Hence we may find ν such that $\delta_{M_\nu} \notin C_{\delta_N}$ and $\delta_{M_\nu} \notin C_{f'(\delta)}$ for any $\delta \in \mathrm{dom}(f')$ above δ_N. There is also $\eta < \delta_{M_\nu}$ such that $[\eta, \delta_{M_\nu}) \cap C_{\delta_N} = \emptyset$ and $[\eta, \delta_{M_\nu}) \cap C_{f'(\delta)} = \emptyset$ for any $\delta \in \mathrm{dom}(f')$ above δ_N.

Now work inside M_ν. By correctness, there is, in M_ν, a condition $(\bar{f}, \bar{b}) \in D$ extending $(f' \restriction \delta_N, b' \restriction \delta_N)$ and such that $\min(\bar{f} \setminus (f' \restriction \delta_n)) > \eta$ (as witnessed by (f', b') itself!).

Finally, $(f' \cup \bar{f}, b' \cup \bar{b})$ is a $\mathcal{P}_{\vec{C}}$–condition extending both (f', b') and $(\bar{f}, \bar{b})$. $\square$

Remark: In above proof, going from (f, b) to $(f \cup \{(\delta_N, \delta_N)\}, b)$ can be seen as implicitly "adding N as side condition".

Note: It follows from the above that PFA implies $\neg$ WCG.

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Finally, $(f' \cup \bar{f}, b' \cup \bar{b})$ is a $\mathcal{P}_{\vec{C}}$–condition extending both (f', b') and $(\bar{f}, \bar{b})$. $\square$

Remark: In above proof, going from (f, b) to $(f \cup \{(\delta_N, \delta_N)\}, b)$ can be seen as implicitly "adding N as side condition".

Note: It follows from the above that PFA implies $\neg$ WCG.

Measuring is the following statement: Suppose $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that each C_δ is a closed subset of δ in the order topology. Then there is a club $D \subseteq \omega_1$ such that for every $\delta \in D$ there is some $\alpha < \delta$ such that either

- $(D \cap \delta) \setminus \alpha \subseteq C_\delta$, or else
- $(D \setminus \alpha) \cap C_\delta = \emptyset$.

We say that D *measures* $\vec{C}$.

• Measuring is equivalent to Measuring restricted to club–sequences.

• Measuring implies $\neg$WCG: Let $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ be a ladder system. Let D be a club measuring $\vec{C}$. Then D' is such that each $\delta \in D'$ has finite intersection with C_δ. Indeed, we can assume that δ is a limit point of D'. But then $D \cap \delta$ cannot have a tail contained in C_δ since it is a limit point of limit points of D and $\mathrm{ot}(C_\delta) = \omega$. Hence $D \cap \delta$ has a tail disjoint from C_δ.

Given a set of ordinals X and an ordinal α say that

- $\text{rank}(X, \alpha) > 0$ iff α is a limit point of ordinals in X, and
- if $\rho > 1$, then $\text{rank}(X, \alpha) \geq \rho$ iff for every $\rho' < \rho$, α is a limit point of ordinals β such that $\text{rank}(X, \beta) \geq \rho'$.

Measuring one club–sequence with finite conditions:

Let $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ club–sequence. Let $\mathcal{P}_{\vec{C}}$ be as follows: Conditions are triples $(f, b, \mathcal{N})$ such that

(1) $f \subseteq \omega_1 \times \mathrm{Lim}(\omega_1)$ is a finite function.

(2) $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and $b(\xi) < f(\xi)$ for each $\xi \in \mathrm{dom}(b)$.

(3) For each $\xi \in \mathrm{dom}(b)$, $C_{f(\xi)} \cap \mathrm{range}(f \restriction \xi) \subseteq b(\xi)$.

(4) $\mathcal{N}$ is a finite $\in$–chain of countable elementary submodels of $H(\omega_2)$.

(5) The following holds for every $\nu \in \mathrm{dom}(f)$.

   (5.1) For every $N \in \mathcal{N}$ such that $\delta_N \leq f(\nu)$ and every club $C \subseteq \omega_1$ in N, $\mathrm{rank}(C, f(\nu)) \geq \nu$.

   (5.2) If $\nu \in \mathrm{dom}(b)$, then for every $N \in \mathcal{N}$ such that $\delta_N \leq f(\nu)$ and every club $C \subseteq \omega_1$ in N, $\mathrm{rank}(C \setminus C_{f(\nu)}, f(\nu)) \geq \nu$.

(6) For every $N \in \mathcal{N}$, $(\delta_N, \delta_N) \in f$.

Extension: $(f_1, b_1, \mathcal{N}_1) \le (f_0, b_0, \mathcal{N}_0)$ iff

- $f_0 \subseteq f_1$,
- $b_0 \subseteq b_1$, and
- $\mathcal{N}_0 \subseteq \mathcal{N}_1$.

$\mathcal{P}_{\vec{C}}$ is proper:

Let $(f, b, \mathcal{N}) \in N$, where $N \prec H(\theta)$ for quite large θ. Let $\delta_N = N \cap \omega_1 \in \omega_1$. Then $(f \cup \{(\delta_N, \delta_N)\}, b, \mathcal{N} \cup \{N \cap H(\omega_2)\})$ is $(N, \mathcal{P}_{\vec{C}})$–generic:

Let $(f', b', \mathcal{N}')$ extend $(f \cup \{(\delta_N, \delta_N)\}, b, \mathcal{N} \cup \{N \cap H(\omega_2)\})$ and let $D \subseteq \mathcal{P}$ dense and open, $D \in N$. By extending $(f', b', \mathcal{N}')$ if necessary we may assume $(f', b', \mathcal{N}') \in D$.

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Note: $f' \restriction \delta_N$, $b' \restriction \delta_N$, $\mathcal{N}' \cap N \in N$. In N pick θ_0 large enough and let $(M_\nu)_{\nu<\omega_1}$ $\subseteq$–continuous chain of countable elementary substructures of $H(\theta_0)$ containing $f' \restriction \delta_N$, $b' \restriction \delta_N$, $\mathcal{N}' \cap N$ and D. Let $C = (\delta_{M_\nu})_{\nu<\omega_1}$.

Assume $\delta_N \in \mathrm{dom}(b')$ (proof in the other case is easier). But then there is some ν such that $\delta_{M_\nu} \notin C_{\delta_N}$ and $\delta_M \notin C_{f(\delta')}$ for any $\delta' \in \mathrm{dom}(b')$ such that $\delta' > \delta_N$ and $b'(\delta') < \delta_N$. By closedness of the C_δ's, there is also $\eta < \delta_M$ such that $[\eta, \delta_M) \cap C_{\delta_N} = \emptyset$ and $[\eta, \delta_M) \cap C_{f'(\delta)} = \emptyset$ for any $\delta \in \mathrm{dom}(f')$ above δ_N such that $b'(\delta') < \delta_N$.

The rest of the proof is now as in the $\neg\,\mathrm{WCG}$ case. $\square$

$\mathcal{P}_{\vec{C}}$ measures $\vec{C}$:

Easy: If G is $\mathcal{P}_{\vec{C}}$–generic and $F_G = \bigcup\{f : (f, b, \mathcal{N}) \in G \text{ for some } b, \mathcal{N}\}$, then $\mathrm{range}(F_G)$ is a club of ω_1 and for each limit ordinal $\delta \in \omega_1$, if $\delta \in \mathrm{dom}(b)$ for some $(f, b, \mathcal{N}) \in G$, then a tail of $\mathrm{range}(F_G)$ is disjoint from $C_{f(\delta)}$.

Now suppose there is no $(f, b, \mathcal{N}) \in G$ such that $\delta \in \mathrm{dom}(b)$. Pick $(f, b, \mathcal{N})$ such that $\delta \in \mathrm{dom}(f)$. We may assume there is $N \in \mathcal{N}$ with $\delta_N \leq \delta$ and a club $C \in N$ such that $\mathrm{rank}(C \setminus C_{f(\delta)}, f(\delta)) = \delta_0 < \delta$. Otherwise we would be able to extend $(f, b, \mathcal{N})$ to $(f, b', \mathcal{N})$ such that $\delta \in \mathrm{dom}(b')$. But then, if $(f', b', \mathcal{N}) \leq (f, b, \mathcal{N})$ and $\delta_0 \in \mathrm{dom}(f')$, $(f', b', \mathcal{N})$ forces that $\mathrm{range}(F_{\dot{G}}) \cap [f'(\delta_0), f(\delta)) \subseteq C_{f(\delta)}$. $\square$

Hence, PFA implies Measuring.

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Now suppose there is no $(f, b, \mathcal{N}) \in G$ such that $\delta \in \text{dom}(b)$. Pick $(f, b, \mathcal{N})$ such that $\delta \in \text{dom}(f)$. We may assume there is $N \in \mathcal{N}$ with $\delta_N \leq \delta$ and a club $C \in N$ such that $\text{rank}(C \setminus C_{f(\delta)}, f(\delta)) = \delta_0 < \delta$. Otherwise we would be able to extend $(f, b, \mathcal{N})$ to $(f, b', \mathcal{N})$ such that $\delta \in \text{dom}(b')$. But then, if $(f', b', \mathcal{N}) \leq (f, b, \mathcal{N})$ and $\delta_0 \in \text{dom}(f')$, $(f', b', \mathcal{N})$ forces that $\text{range}(F_{\dot{G}}) \cap [f'(\delta_0), f(\delta)) \subseteq C_{f(\delta)}$. $\square$

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Hence, PFA implies Measuring.

We may consider the following family of strengthenings of Measuring.

Definition

Given a cardinal κ, Measuring_κ holds if and only if for every family $\mathcal{C}$ consisting of closed subsets of ω_1 such that $|\mathcal{C}| \leq \kappa$ there is a club $D \subseteq \omega_1$ with the property that for every $\delta \in D$ and every $C \in \mathcal{C}$ there is some $\alpha < \delta$ such that either

- $(D \cap \delta) \setminus \alpha \subseteq C$, or
- $((D \cap \delta) \setminus \alpha) \cap C = \emptyset$.

- Measuring_λ implies Measuring_κ whenever $\lambda < \kappa$
- $\text{Measuring}_{\aleph_0}$ is true in ZFC.
- $\text{Measuring}_{\aleph_1}$ implies Measuring.

Recall that the splitting number, $\mathfrak{s}$, is the minimal cardinality of a splitting family, i.e., of a collection $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ such that for every $Y \in [\omega]^{\aleph_0}$ there is some $X \in \mathcal{X}$ such that $X \cap Y$ and $Y \setminus X$ are both infinite.

Fact

Measuring$_{\mathfrak{s}}$ is false.

Proof.

Let $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ be a splitting family. Let $(C_\delta)_{\delta \in \mathrm{Lim}(\omega)}$ be a ladder system on ω_1 and let $\mathcal{C}$ be the collection of all sets of the form

$$Z_\delta^X = \bigcup \{[C_\delta(n),\ C_\delta(n+1)] \ :\ n \in X\} \cup \{\delta\}$$

for some $\delta \in \mathrm{Lim}(\omega_1)$ and $X \in \mathcal{X}$. Let D be a club of ω_1, let $\delta < \omega_1$ be a limit point of D, and let $Y = \{n < \omega \ :\ [C_\delta(n),\ C_\delta(n+1)] \cap D \neq \emptyset\}$. Let $X \in \mathcal{X}$ be such that $X \cap Y$ and $Y \setminus X$ are infinite. Then $Z_\delta^X \cap D$ and $D \setminus Z_\delta^X$ are both cofinal in δ. Hence, D does not measure $\mathcal{C}$. $\square$

The following question is open.

Question

Is $\text{Measuring}_{\aleph_1}$ consistent?

Iterated forcing with side conditions

Recall our problem: Iterate (interesting) non–c.c.c. proper forcing while getting $2^{\aleph_0} > \aleph_2$ in the end.

Neither countable supports, nor uncountable supports nor finite supports work.

A solution: Use finite supports, together with countable elementary substructures of some $H(\theta)$ as side conditions affecting the whole iteration or initial segments of the iteration in order to ensure properness. As mentioned, the idea of using countable structures as side conditions in order to “force” a non–proper forcing to become proper is old. However, the idea of doing this in the context of actual iterations is relatively new.

Typically we will want our iteration to have the $\aleph_2$–c.c. (after all we are interested in $2^{\aleph_0}$ arbitrarily large).The natural approach of using finite $\in$–chains of structures won’t work, though, since we have too many structures and would therefore lose the $\aleph_2$–c.c. We will replace $\in$–chains of structures by “matrices” of structures with suitable symmetry properties. If we start with CH and consider only iterands with the $\aleph_2$–c.c., we may succeed.

Symmetric systems of elementary substructures

Definition

Let θ be a cardinal and $T \subseteq H(\theta)$ (such that $\bigcup T = H(\theta)$). A finite set $\mathcal{N} \subseteq [H(\theta)]^{\aleph_0}$ is a T*–symmetric system* iff the following holds for all N, N_0, $N_1 \in \mathcal{N}$:

(1) $(N; \in, Y) \preccurlyeq (H(\theta); \in, T)$

(2) If $\delta_{N_0} = \delta_{N_1}$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0; \in, T) \longrightarrow (N_1; \in, T)$$

Furthermore, Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) If $\delta_{N_0} = \delta_{N_1}$ and $N \in N_0 \cap \mathcal{N}$, then $\Psi_{N_0,N_1}(N) \in \mathcal{N}$.

(4) If $\delta_{N_0} < \delta_{N_1}$, then there is some $N_1' \in \mathcal{N}$ such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.

- Symmetric systems had previously been considered in (at least) work of Todorčević, Abraham–Cummings and Koszmider. Again, not in the context of forcing iterations.
- The def. of symmetric system guarantees that

(4)' if $N_0, N_1 \in \mathcal{N}$ and $\delta_{N_0} < \delta_{N_1}$, then there is some $N_0' \in N_1 \cap \mathcal{N}$ such that $\delta_{N_0'} = \delta_{N_0}$ and $N_0 \cap N_1 = N_0 \cap N_0'$.

(In fact, $N_0' = \Psi_{N_1', N_1}(N_0)$, where $N_1' \in \mathcal{N}$ is such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.) This property is important in many applications. Sometimes it is enough to keep (1)–(3) and weaken (4) to (4)'. The resulting object is called *partial T–symmetric system*.

Two amalgamation lemmas

1st amalgamation lemma: If $\mathcal{N}$ and $\mathcal{N}'$ are T–symmetric systems, $(\bigcup \mathcal{N}) \cap (\bigcup \mathcal{N}') = X$, and there are enumerations $(N_i)_{i<n}$ and $(N'_i)_{i<n}$ of $\mathcal{N}$, $\mathcal{N}'$, resp., for which there is an isomorphism

$$\Psi : (\bigcup \mathcal{N}; \in, N_i, T, X)_{i<n} \longrightarrow (\bigcup \mathcal{N}'; \in, N'_i, T, X)_{i<n}$$

then $\mathcal{N} \cup \mathcal{N}'$ is a T–symmetric system.

2nd amalgamation lemma: Let $\mathcal{N}$ be a T–symmetric system and $M \in \mathcal{N}$. Suppose $\mathcal{M} \in M$ is a T–symmetric system such that $\mathcal{N} \cap M \subseteq \mathcal{M}$. Let

$$\mathcal{N}^M(\mathcal{M}) = \mathcal{N} \cup \{\Psi_{M,M'}(N) \ : \ N \in \mathcal{M},\ M' \in \mathcal{N} \ : \ \delta_{M'} = \delta_M\}$$

Then $\mathcal{N}^M(\mathcal{M})$ is a T–symmetric system.

Corollaries Let

$$\mathrm{Symm}_T = (\{\mathcal{N} : \mathcal{N}\ T\text{–symmetric system}\}, \supseteq)$$

Using 1st amalgamation lemma:

Corollary 1 (CH) Symm_T is $\aleph_2$–Knaster.

Using 2nd amalgamation lemma:

Corollary 2 Symm_T is strongly proper (i.e., for all $p \in N$ there is $q \leq p$ such that for every $q' \leq q$, there is some $\pi(q') \in \mathcal{P} \cap N$ such that every $r \in N$ such that $r \leq \pi(q')$ is compatible with q').

Using Corollary 2 and the proof of Corollary 1:

Corollary 3 (CH) Symm_T adds new reals but preserves CH. In fact, Symm_T
adds exactly $\aleph_1$–many reals, all of which are Cohen reals over V.

Proof: Suppose, towards a contradiction, there is $\mathcal{N}$ and a sequence $(\dot{r}_\alpha : \alpha < \omega_2)$ of nice names for subsets of ω such that $\mathcal{N} \Vdash \dot{r}_\alpha \neq \dot{r}_{\alpha'}$ for all $\alpha \neq \alpha'$. For each α let $N_\alpha = N^*_\alpha \cap H(\theta)$ for some countable $N^*_\alpha \prec H(\chi)$ (χ larger) containing T, $\mathcal{N}$ and $\dot{r}_\alpha$. By CH there are $\alpha \neq \alpha'$ such that

$$(N_\alpha, \in, T, \dot{r}_\alpha) \cong (N_{\alpha'}, \in, T, \dot{r}_{\alpha'})$$

and $\Psi_{N_\alpha, N_{\alpha'}}$ fixes $N_\alpha \cap N_{\alpha'}$. Let $\mathcal{M} = \mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$.

Then $\mathcal{M} \Vdash \dot{r}_\alpha = \dot{r}_{\alpha'}$:

Let $\mathcal{N}' \leq \mathcal{M}$ and $n \in \omega$ such that $\mathcal{N} \Vdash n \in \dot{r}_\alpha$. By $(\mathrm{Symm}_T, N^*_\alpha)$–genericity of $\mathcal{M}$, there is some $\mathcal{Q} \in \mathrm{Symm}_T \cap N_\alpha$ such that $\mathcal{Q}$ is in the antichain of $\dot{r}_\alpha$ forcing $n \in \dot{r}_\alpha$. Since

$$\Psi_{N_\alpha, N_{\alpha'}} : (N_\alpha, \in, T, \dot{r}_\alpha) \longrightarrow (N_{\alpha'}, \in, T, \dot{r}_{\alpha'})$$

is an isomorphism, $\Psi_{N_\alpha, N_{\alpha'}}(\mathcal{Q}) \in \mathrm{Symm}_T \cap N_{\alpha'}$ is in the antichain of $\dot{r}_{\alpha'}$ forcing $n \in \dot{r}_{\alpha'}$. But by symmetry, $\mathcal{N}'$ extends $\Psi_{N_\alpha, N_{\alpha'}}(\mathcal{Q})$.

This shows $\mathcal{M} \Vdash \dot{r}_\alpha \subseteq \dot{r}_{\alpha'}$, and by arguing symmetrically we show $\mathcal{M} \Vdash \dot{r}_{\alpha'} \subseteq \dot{r}_\alpha$. $\square$

Iterating: A typical construction.

Start with CH, let κ regular with $2^{<\kappa} = \kappa$. Fix suitable $T \subseteq H(\kappa)$. Let $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ be such that for all α, a condition in $\mathcal{P}_\alpha$ is a pair $q = (F, \Delta)$ such that:

(1) F is a finite function such that $\mathrm{dom}(F) \subseteq \alpha$ ($\mathrm{dom}(F)$ is the support of q).

(2) Δ is a finite set of pairs (N, γ), where $N \in [H(\kappa)]^{\aleph_0}$, $\gamma \leq \alpha$, $\gamma \leq \sup(N \cap \kappa)$, and where $\mathrm{dom}(\Delta)$ is a (partial) T–symmetric system (γ is the *marker* associated to N).

(3) For all $\beta < \alpha$,

$$q|_\beta := (F \restriction \beta, \{(N, \min\{\gamma, \beta\}) : (N, \gamma) \in \Delta\})$$

is a $\mathcal{P}_\beta$–condition.

(4) For every $\xi \in \mathrm{dom}(F)$,

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \in \Phi^*(\xi)$$

where $\Phi^*(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing, and $\Phi^*(\xi) = \Phi(\xi)$ if $\Phi(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing (and where Φ is a suitable bookkeeping function on κ).

(5) For every $\xi \in \mathrm{dom}(F)$ and every $(N, \gamma) \in \Delta$, if $\xi < \gamma$ and $\xi \in N$, then

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \text{ is } (N[\dot{G}_\xi], \Phi^*(\xi))\text{–generic}$$

Given $\mathcal{P}_\alpha$–conditions $q_0 = (F_0, \Delta_0)$, $q_1 = (F_1, \Delta_1)$, $q_1 \leq_\alpha q_0$ iff

(a) for every $(N, \gamma) \in \Delta_0$ there is some $\gamma' \geq \gamma$ such that $(N, \gamma') \in \Delta_1$,

(b) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$, and

(c) for every $\xi \in \mathrm{dom}(F_0)$,

$$q_0|_\xi \Vdash_{\mathcal{P}_\xi} F_1(\xi) \leq_{\Phi^*(\xi)} F_0(\xi)$$

This way it is for example possible to build models of forcing axioms for classes Γ such that $\{\mathbb{P} : \mathbb{P} \text{ c.c.c.}\} \subseteq \Gamma \subseteq \{\mathbb{P} : \mathbb{P} \text{ proper}\}$ together with $2^{\aleph_0} > \aleph_2$.

[More of this later.]

Measuring together with $2^{\aleph_0} > \aleph_2$

Theorem

(A.–Mota (JSL 2017, to appear)) (CH) Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*

(2) *$\mathcal{P}$ forces the following statements.*

- *Measuring*
- *$2^\mu = \kappa$ for every infinite cardinal $\mu < \kappa$.*

This theorem answers a question of J. Moore, who asked if Measuring is compatible with $2^{\aleph_0} > \aleph_2$.

Proof of the main theorem

Yet another notion of rank: Given sets N, $\mathcal{X}$ and an ordinal ρ, we define $\mathrm{rank}(\mathcal{X}, N) \geq \rho$ recursively by:

- $\mathrm{rank}(\mathcal{X}, N) \geq 1$ if and only if for every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$.
- If $\rho > 1$, then $\mathrm{rank}(\mathcal{X}, N) \geq \rho$ if and only if for every $\rho' < \rho$ and every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$ and $\mathrm{rank}(\mathcal{X}, M) \geq \rho'$.

Let $\Phi : \kappa \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is unbounded in κ for all $x \in H(\kappa)$. Notice that Φ exists by $2^{<\kappa} = \kappa$. Let also $\lhd$ be a well–order of $H((2^{\kappa})^{+})$.

Let $(\theta_\alpha)_{\alpha<\kappa}$ be the sequence of cardinals defined by $\theta_0 = |H((2^\kappa)^+)|^+$ and $\theta_\alpha = (2^{<\sup_{\beta<\alpha}\theta_\beta})^+$ if $\alpha > 0$.

For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $(\theta_\beta)_{\beta<\alpha}$, and let

$$\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \,:\, N^* \in \mathcal{M}^*_\alpha\}$$

Let T^α be the $\lhd$–first $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N, \in, T \cap N) \prec (H(\kappa), \in, T)$, then $N \in \mathcal{M}_\alpha$.

Let also

$$\mathcal{T}^\alpha = \{N \in [H(\kappa)]^{\aleph_0} \,:\, (N, \in, T^\alpha \cap N) \prec (H(\kappa), \in, T^\alpha)\}.$$

Fact

Let $\alpha < \beta \leq \kappa$.

1. *If $N^* \in \mathcal{M}^*_\beta$ and $\alpha \in N^*$, then $\mathcal{M}^*_\alpha \in N^*$ and $N^* \cap H(\kappa) \in \mathcal{T}^\alpha$.*
2. *If $N, N' \in \mathcal{T}^\beta$, $\Psi : (N, \in, T^\beta \cap N) \longrightarrow (N', \in, T^\beta \cap N')$ is an isomorphism, and $M \in N \cap \mathcal{T}^\beta$, then $\Psi(M) \in \mathcal{T}^\beta$.*

Our forcing $\mathcal{P}$ will be $\mathcal{P}_\kappa$, where $(\mathcal{P}_\beta : \beta \leq \kappa)$ is the sequence of posets to be defined next.

In the following definition, and throughout the lectures, if q is an ordered pair (F, Δ), we will denote F and Δ by F_q and Δ_q, respectively.

Let $\beta \leq \kappa$ and suppose $\mathcal{P}_\alpha$ has been defined for all $\alpha < \beta$. Conditions in $\mathcal{P}_\beta$ are ordered pairs $q = (F, \Delta)$ with the following properties.

(1) F is a finite function with $\mathrm{dom}(F) \subseteq \beta$.

(2) Δ is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$ and γ is an ordinal such that $\gamma \leq \beta$ and $\gamma \leq \sup(N \cap \kappa)$.

(3) $\mathcal{N}^q_\beta := \{N : (N, \beta) \in \Delta,\ \beta \in N\}$ is a T^β–symmetric system.

(4) For every $\alpha < \beta$, the restriction of q to α,

$$q|_\alpha := (F \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}),$$

is a condition in $\mathcal{P}_\alpha$.

(5) Suppose $\beta = \alpha + 1$. Let $\mathcal{N}^{\dot{G}_\alpha}$ be a $\mathcal{P}_\alpha$–name for $\bigcup\{\mathcal{N}^r_\alpha : r \in \dot{G}_\alpha\}$ (where $\dot{G}_\alpha$ is the canonical $\mathcal{P}_\alpha$–name for the generic object). Let $\dot{C}^\alpha$ be a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1 such that $\mathcal{P}_\alpha$ forces that

- $\dot{C}^\alpha = \Phi(\alpha)$ in case $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1, and that
- $\dot{C}^\alpha$ is some fixed club–sequence on ω_1 in the other case.

If $\alpha \in \mathrm{dom}(F)$, then $F(\alpha) = (f, b, \mathcal{O})$ has the following properties.

(a) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.

(b) $\mathcal{O} \subseteq \mathcal{N}^{q|\alpha}_\alpha$ is a T^β–symmetric system.

(c) $\mathrm{range}(f) \subseteq \{\delta_N : N \in \mathcal{O}\}$

(d) For every $\delta \in \mathrm{dom}(f)$, if $N \in \mathcal{O}$ is such that $p(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta, N) \geq \delta$$

(e) $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and $b(\delta) < f(\delta)$ for every $\delta \in \mathrm{dom}(b)$.

(f) For every $\delta \in \mathrm{dom}(b)$,

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{range}(f \restriction \delta) \cap \dot{C}^\alpha(f(\delta)) \subseteq b(\delta)$$

(g) For every $\delta \in \mathrm{dom}(b)$, if $N \in \mathcal{O}$ is such that $f(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta : \delta_M \notin \dot{C}^\alpha(f(\delta))\}, N) \geq \delta$$

(h) If $N \in \mathcal{N}^q_\beta$, then $N \in \mathcal{O}$, $\delta_N \in \mathrm{dom}(f)$ and $f(\delta_N) = \delta_N$.

Given $\mathcal{P}_\beta$–conditions $q_i = (F_i, \Delta_i)$, for $i = 0, 1$, q_1 extends q_0 if and only if

- $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and for all $\alpha \in \mathrm{dom}(F_0)$, if $F_0(\alpha) = (f, b, \mathcal{O})$ and $F_1(\alpha) = (f', b', \mathcal{O}')$, then $f \subseteq f'$, $b \subseteq b'$ and $\mathcal{O} \subseteq \mathcal{O}'$, and
- $\Delta_0 \subseteq \Delta_1$

Lemma

Let $\alpha \leq \beta \leq \kappa$. If $q = (F_q, \Delta_q) \in \mathcal{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathcal{P}_\beta$, and $q \leq_\alpha r|_\alpha$, then

$$r \wedge_\alpha q := (F_q \cup (F_r \restriction [\alpha,\ \beta)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\beta$ extending r. Hence, $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

Proof.

This is thanks to the use of the markers γ in the (N, γ)'s from Δ.

Lemma

For every ordinal $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

Proof.

A standard Δ–system argument using CH. □

Say that q *is* $(N, \mathcal{P}_\alpha)$*–pre-generic* iff $(N, \alpha) \in \Delta_q$ and $\alpha \in N$.

A technical lemma:

Lemma

Let $\beta < \kappa$. Suppose q is $(M, \mathcal{P}_\beta)$–generic whenever q is $(M, \mathcal{P}_\beta)$–pre-generic and $M \in \mathcal{T}^{\beta+1}$.[1] Then for every $R \subseteq H(\kappa)$, if M is such that $\langle M, T^{\beta+1}, R\rangle \prec \langle H(\kappa), T^{\beta+1}, R\rangle$, then $\mathcal{P}_\beta$ forces that if $M \in \mathcal{N}^{\dot{G}_\beta}$, then $\langle M[\dot{G}_\beta], \dot{G}_\beta, R\rangle \prec \langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R\rangle$.

[1] We will see, in the following lemma, that this hypothesis is true.

Properness lemma:

Lemma

Suppose $\alpha < \kappa$ and $N \in \mathcal{T}^{\alpha+1}$. Then the following holds.

$(1)_\alpha$ *For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$–pre-generic.*

$(2)_\alpha$ *If $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$–pre-generic, then q is $(N, \mathcal{P}_\alpha)$–generic.*

Proof of the lemma on the board.

Lemma

$\mathcal{P}_\kappa$ forces Measuring.

Proof: Let $\alpha < \kappa$, let G be $\mathcal{P}_\alpha$–generic, and suppose $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1. Let $\vec{C} = \Phi(\alpha)_G = (C_\epsilon : \epsilon \in \mathrm{Lim}(\omega_1))$. Let H be a $\mathcal{P}_{\alpha+1}$–generic filter such that $H \restriction \mathcal{P}_\alpha = G$, and let $D = \bigcup \mathrm{range}\{f_{q,\alpha} : q \in H\}$. By the $\aleph_2$–c.c. of $\mathcal{P}_\kappa$ and the choice of Φ, the conclusion will follow, by standard arguments, if we show that D is a club of ω_1 measuring $\vec{C}$.

By standard density arguments, D is a club of ω_1. Also, if $\epsilon \in D$ is such that there is some $q \in H$ such that $\epsilon = f_{q,\alpha}(\delta)$ for some $\delta \in \mathrm{dom}(b_{q,\alpha})$, then a tail of $D \cap \epsilon$ is disjoint from C_ϵ. Hence, it suffices to show that if $\delta \in \omega_1$ is such that $\delta \notin \mathrm{dom}(b_{q,\alpha})$ for every $q \in H$ and ϵ is such that $f_{q,\alpha}(\delta) = \epsilon$ for some $q \in H$, then a tail of $D \cap \epsilon$ is contained in C_ϵ.

But this implies that there is some $q \in H$ and some $N \in \mathcal{O}_{q,\alpha}$ such that $f_{q,\alpha}(\delta) = \delta_N$ and such that

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \text{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} \,:\, \delta_M \notin \Phi(\alpha)(\epsilon)\}, N) = \delta_0$$

for some given $\delta_0 < \delta$. It will now be enough to find some $\eta \in [\delta_0,\, \delta)$ and some extension q^* of q such that every extension q' of q^* is such that $q'|_\alpha$ forces that $f_{q',\alpha}(\delta') \in \Phi(\alpha)(\delta)$ for every $\delta' \in \text{dom}(f_{q',\alpha}) \cap [\eta,\, \delta)$.

By extending $q|_\alpha$ if necessary we may assume that there is some $a \in N$ such that $q|_\alpha$ forces that if $M \in N \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ is such that $a \in M$ and $\text{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) \geq \delta_0$, then $\delta_M \in \Phi(\alpha)(\epsilon)$.

Again by extending $q|_\alpha$ if necessary, we may also assume that there is some $M \in N \cap \mathcal{N}_\alpha^{q|_\alpha} \cap \mathcal{T}^{\alpha+1}$ containing all relevant objects, where this includes a, and such that $q|_\alpha$ forces $\text{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta_1$, where $\delta_1 < \delta$ is such that $\delta_1 > \max(\text{dom}(f_{q,\alpha} \restriction \delta))$ and $\delta_1 \geq \delta_0$. Let now q^* be any extension of q such that $M \in \mathcal{O}_{q^*,\alpha}$ and such that $f_{q^*,\alpha}(\delta_1) = \delta_M$. Now it is easy to verify that $\eta = \delta_1$ and q^* are as desired.

Indeed, it suffices to prove that if q' is any condition extending q^* and $R \in \mathcal{O}_{q',\alpha}$ is such that $\delta_R > \delta_M$ and $\delta_R < \delta_N$, then $q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R \in \Phi(\alpha)(\epsilon)$. But by symmetry of $\mathcal{O}_{q',\alpha}$ there is some $R' \in \mathcal{O}_{q',\alpha} \cap N$ such that $M \in R'$ and $\delta_{R'} = \delta_R$. Since $a \in R'$ and $q'|_\alpha$ extends $q^*|_\alpha$, it follows then that $q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R = \delta_{R'} \in \Phi(\alpha)(\epsilon)$. $\square$

This finishes the proof of the theorem.

Building models of CH

The project of building models of consequences of forcing axioms like PFA together with CH has a long history, starting with:

Theorem

(Jensen) It is consistent to have CH together with the nonexistence of Suslin trees.

The proof of these results usually proceed by showing that some CS iteration of proper forcing notion not adding reals does not add reals at limit stages. A lot of quite technical work in this direction, especially due to Shelah.
Strongest results so far in this direction for negations of $\diamondsuit$ are in the region of the relative consistency of $\neg$ WCG with CH (Shelah, NNR revisited).

There are well–known limitations to this line of work (i.e., countable support iterations of proper forcing not adding reals may add reals at limit stages). Example:

Uniformization holds iff for every club–sequence $\vec{C} = (C_\delta\,:\,\delta \in \mathrm{Lim}(\omega_1))$ and every $F : \mathrm{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is $G : \omega_1 \longrightarrow \{0, 1\}$ such that for every $\delta \in \mathrm{Lim}(\omega_1)$, $F(\delta) = G(\beta)$ for all β on a tail of C_δ.

The natural forcing for, given $\vec{C}$ and F, adding a uniformising function by initial segments is proper and does not add new reals.

However,

Fact

(Shelah?) Uniformization implies $2^{\aleph_0} = 2^{\aleph_1}$*.*

Proof.

Fix a bijection $h : \omega \longrightarrow \omega \times \omega$ such that $i, j \leq n$ whenever $h(n+1) = (i, j)$. For each $f : \omega_1 \longrightarrow 2$ construct functions $g_n : \omega_1 \longrightarrow 2$ such that $g_0 = f$ and

$$g_{n+1} \restriction C_\delta \equiv_{fin} g_i(\delta + j)$$

whenever δ is a limit ordinal and $h(n+1) = (i, j)$. Given g_k $(k \leq n)$, g_{n+1} exists by applying Uniformization to the colouring

$$\delta \longmapsto g_i(\delta + j)$$

where $h(n+1) = (i, j)$. But then, for each limit ordinal $\delta \geq \omega$, $(g_n \restriction \delta)_{n<\omega}$ uniquely determines $(g_n \restriction \delta + \omega)_{n<\omega}$. In particular, $(g \restriction \omega)_{n<\omega}$ uniquely determines $(g_n)_{n<\omega}$ and hence $g_0 = f$. □

The following is an important test question in this context.

Question (J. Moore) Is Measuring consistent with CH?

Few new reals

Recall: $\mathrm{Symm}_T = (\{\mathcal{N} : \mathcal{N}\ T\text{–symmetric system}\}, \supseteq)$ adds new reals but preserves CH.

Now: Suppose we build an iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ with symmetric systems of models as side conditions and we require that q extends $\Psi_{N,N'}(q|_\alpha \restriction N)$ whenever (N, γ), $(N', \gamma') \in \Delta_q$, $\delta_N = \delta_{N'}$, $\alpha \in N \cap (\gamma + 1)$ and $\Psi_{N,N'}(\alpha) \leq \gamma'$. Then the same proof for Symm_T should show that $\mathcal{P}_\kappa$ preserves CH (although it adds new reals).

We will have to tinker a bit with this idea before this leads to an iteration doing something interesting.

Work in progress (A.–Mota) (CH) Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There should then be a partial order $\mathcal{P}$ with the following properties.

(1) $\mathcal{P}$ is proper and $\aleph_2$–Knaster.

(2) $\mathcal{P}$ forces the following statements.

   (a) Measuring
   (b) CH
   (c) $2^{\mu} = \kappa$ for every uncountable cardinal $\mu < \kappa$.

Sketch of the construction:

Let $\Phi : \kappa \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is unbounded in κ for all $x \in H(\kappa)$. Let also $\lhd$ be a well–order of $H((2^\kappa)^+)$.
Let $(\theta_\alpha)_{\alpha<\kappa}$ be the sequence of cardinals defined by $\theta_0 = |H((2^\kappa)^+)|^+$ and $\theta_\alpha = (2^{<\sup_{\beta<\alpha}\theta_\beta})^+$ if $\alpha > 0$. For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $(\theta_\beta)_{\beta<\alpha}$, and let $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \,:\, N^* \in \mathcal{M}^*_\alpha\}$. Let T_α be the $\lhd$–first $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N; \in, T) \prec (H(\kappa); \in, T)$, then $N \in \mathcal{M}_\alpha$. Let also

$$\mathcal{T}_\alpha = \{N \in [H(\kappa)]^{\aleph_0} \,:\, (N; \in, T_\alpha) \prec (H(\kappa); \in, T_\alpha)\}$$

and

$$\vec{T}_\alpha = \{(a, \xi) \in H(\kappa) \times \alpha + 1 \,:\, a \in T_\xi\}$$

Let $\beta \leq \kappa$ and suppose $\mathcal{P}_\alpha$ defined for all $\alpha < \beta$.

A triple $q = (F, \Delta, \tau)$ is called a $\mathcal{P}_\beta$–*pre-condition* if and only if it has the following properties.

(1) F is a function with finite support such that $\text{dom}(F) = \beta$ and such that $F(\alpha)$ is a triple $(f, b, \mathcal{O})$ for every $\alpha \in \text{dom}(F)$.

(2) Δ is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$, $\gamma \in \text{cl}(N \cap \text{Ord})$ and $\gamma \leq \beta$.

(3) τ is an equivalence relation on Δ such that $\delta_N = \delta_{N'}$ whenever $((N, \gamma), (N', \gamma')) \in \tau$ for some γ and γ'.

(4) $\mathcal{N}^q_\beta$ is a T_β–symmetric system.

(5) For every $\alpha < \beta$, the restriction of q to α,

$$q|_\alpha := (F \restriction \alpha, \Delta|_\alpha, \tau|_\alpha),$$

is a condition in $\mathcal{P}_\alpha$.

(6) Fix $\alpha < \beta$.
Let $\dot{C}^\alpha$ be a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1 such that $\mathcal{P}_\alpha$ forces that

- $\dot{C}^\alpha = \Phi(\alpha)$ in case $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1, and that
- $\dot{C}^\alpha$ is some fixed club–sequence on ω_1 in the other case.

If $\alpha \in \mathrm{supp}(F)$, then $F(\alpha) = (f^q_\alpha, b^q_\alpha, \mathcal{O}^q_\alpha)$ has the following properties.

(a) $f^q_\alpha \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.
(b) $\mathcal{O}^q_\alpha \subseteq \mathcal{N}^{q|_\alpha}_\alpha$ is a $T_{\alpha+1}$–symmetric system.
(c) $\mathrm{range}(f^q_\alpha) \subseteq \{\delta_N : N \in \mathcal{O}^q_\alpha\}$
(d) $\mathrm{dom}(b^q_\alpha) \subseteq \mathrm{dom}(f^q_\alpha)$ and $b^q_\alpha(\delta) < f^q_\alpha(\delta)$ for every $\delta \in \mathrm{dom}(b^q_\alpha)$.
(e) For every $\delta \in \mathrm{dom}(b^q_\alpha)$,

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{range}(f^q_\alpha \restriction \delta) \cap \dot{C}^\alpha(f^q_\alpha(\delta)) \subseteq b^q_\alpha(\delta)$$

(f) If $N \in \mathcal{N}^q_{\alpha+1}$, then $N \in \mathcal{O}^q_\alpha$, $\delta_N \in \mathrm{dom}(f^q_\alpha)$ and $f^q_\alpha(\delta_N) = \delta_N$.

(7) For all (N, γ), $(N', \gamma') \in \Delta$ such that $(N, \gamma)\, \tau\, (N', \gamma')$ there is some $n < \omega$ such that
$n = |\operatorname{dom}(F) \cap N \cap \min\{\gamma, \Psi_{N',N}(\gamma')\}| =$
$|\operatorname{dom}(F) \cap N' \cap \min\{\gamma', \Psi_{N,N'}(\gamma)\}|$; furthermore, letting $(\alpha_i)_{i<n}$ and $(\alpha'_i)_{i<n}$ be the strictly increasing enumerations of $\operatorname{supp}(F) \cap N \cap \min\{\gamma, \Psi_{N',N}(\gamma')\}$ and $\operatorname{supp}(F) \cap N' \cap \min\{\gamma', \Psi_{N,N'}(\gamma)\}$, respectively, $\Psi_{N,N'}$ is an isomorphism between the structures

$$(N; \in, \Phi, \vec{T}_{\min\{\gamma, \Psi_{N',N}(\gamma')\}}, \Delta, f^q_{\alpha_i}, b^q_{\alpha_i}, \mathcal{O}^q_{\alpha_i})_{i<n}$$

and

$$(N; \in, \Phi, \vec{T}_{\min\{\gamma', \Psi_{N,N'}(\gamma)\}}, \Delta, f^q_{\alpha'_i}, b^q_{\alpha'_i}, \mathcal{O}^q_{\alpha'_i})_{i<n}$$

Also, given $\mathcal{P}_\beta$–pre-conditions q_i, for $i = 0, 1$, let us say that q_1 extends q_0 if and only if

1. $\mathrm{supp}(F_{q_0}) \subseteq \mathrm{supp}(F_{q_1})$ and for all $\alpha \in \mathrm{supp}(F_{q_0})$, $f_\alpha^{q_0} \subseteq f_\alpha^{q_1}$, $b_\alpha^{q_0} \subseteq b_\alpha^{q_1}$ and $\mathcal{O}_\alpha^{q_0} \subseteq \mathcal{O}_\alpha^{q_1}$,
2. $\Delta_{q_0} \subseteq \Delta_{q_1}$, and
3. $\tau_{q_0} \subseteq \tau_{q_1}$

Let us denote by $\mathcal{P}_\beta^0$ the collection of $\mathcal{P}_\beta$–pre-conditions ordered by the above extension relation.

We are now ready to define $\mathcal{P}_\beta$.

$\mathcal{P}_\beta$ is the suborder of $\mathcal{P}_\beta^0$ consisting of all those $q = (F, \Delta, \tau) \in \mathcal{P}_\beta^0$ with the property that, if $\beta = \alpha_0 + 1$, then for every $\alpha \in \mathrm{supp}(F)$ and $\delta \in \mathrm{dom}(f_\alpha^q)$, if $N \in \mathcal{O}_\alpha^q$ is such that $f_\alpha^q(\delta) = \delta_N$, then

$$q|_{\alpha_0} \Vdash_{\mathcal{P}_{\alpha_0}^q} \mathrm{rank}(\mathcal{X}_{\delta_N}^\alpha, N) \geq \delta,$$

where

(i) $\mathcal{P}_{\alpha_0}^q$ is the suborder of $\mathcal{P}_{\alpha_0}$ consisting of all those $p \in \mathcal{P}_{\alpha_0}$ such that $F_{\mathrm{Symm}(p \oplus q[\alpha_0])}(\alpha) = F_q(\alpha)$, and

(ii) $\mathcal{X}^{\alpha}_{\delta_N}$ is the set of $M \in \mathcal{N}_{\alpha}^{\dot{G}_{\mathcal{P}^q_{\alpha_0}}} \cap \mathcal{T}_{\alpha+1}$ such that $\delta_M \notin \dot{C}^{\xi}(\delta')$ for all $\delta' \geq \delta$ and all $\xi \in \mathcal{W}^{\alpha}_{\delta_N}$ such that $\delta' \in \mathrm{dom}(b^p_{\xi})$ and $b^p_{\xi}(\delta') \leq \delta_M$ for some $p \in \dot{G}_{\mathcal{P}^0_{\alpha}} \cup \{q\}$, and where, for every ordinal η, $\mathcal{W}^{\alpha}_{\eta}$ is the union of $\{\alpha\}$ and the set of ordinals of the form $\Psi_{Q,Q'}(\alpha)$, where, for some $p \in \dot{G}_{\mathcal{P}^q_{\alpha_0}} \cup \{q\}$,

- $(Q, \gamma)\, \tau_p\, (Q', \gamma')$,
- $\eta \leq \delta_Q$,
- $\alpha \in Q$,
- $\alpha < \gamma$, and
- $\Psi_{Q,Q'}(\alpha) < \gamma'$.

Why shouldn't we be able to use this machinery to force Uniformization together with CH? (We know that's impossible since Uniformization implies $2^{\aleph_0} = 2^{\aleph_1}$.)

The reason boils down to the fact that for Uniformization, given $\vec{C}$ and $F : \mathrm{Lim}(\omega_1) \longrightarrow 2$, we are required for the uniformising function $G : \omega_1 \longrightarrow 2$ to be such that $G(\delta) \restriction C_\delta \equiv_{fin} F(\delta)$, whereas for Measuring we have (relative) freedom to opt for a tail of $D \cap \delta$ to be contained in or disjoint from C_δ. Because of this, the construction does in fact break down if you want to do it for Uniformization instead of Measuring.

Forcing axioms

Definition

A partial order $\mathbb{P}$ has the $\aleph_{1.5}$–c.c. iff for every $\theta \geq |\mathbb{P}|$ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite collection $\mathcal{N}$ of countable $N \prec H(\theta)$ such that $\mathbb{P} \in N$ and every $p \in \mathbb{P}$, if $p \in N$ for some $N \in \mathcal{N}$ with δ_N minimal among $\{\delta_M : M \in \mathcal{N}\}$, then there is some $q \leq p$ such that q is $(N, \mathbb{P})$–generic for all $N \in \mathcal{N}$.

Clearly, $\{\mathbb{P} : \mathbb{P} \text{ c.c.c.}\} \subseteq \{\mathbb{P} : \mathbb{P} \text{ has the } \aleph_{1.5}\text{–c.c.}\} \subseteq \{\mathbb{P} : \mathbb{P} \text{ proper and has the } \aleph_2\text{–c.c.}\}$.

Let's see the second inclusion: Suppose $\mathbb{P}$ has the $\aleph_{1.5}$–c.c. Then $\mathbb{P}$ is clearly proper. Suppose A is a maximal antichain of $\mathbb{P}$ such that $|A| \geq \aleph_2$. Let θ and D witness the $\aleph_{1.5}$–c.c. of $\mathbb{P}$ and let N_p be, for every $p \in A$, a member of D such that $A, p \in N_p$. Let $\delta < \omega_1$ be such that the set A' of $p \in A$ such that $\delta_{N_p} = \delta$ is uncountable. Pick some $p_0 \in A'$ and some $p_1 \in A' \setminus N_{p_0}$. Then, since $\delta_{N_{p_1}} = \delta_{N_{p_0}} = \delta$ and $p_1 \in N_{p_1}$, there is a condition $q \leq p_1$ such that q is $(N_{p_0}, \mathbb{P})$–generic. In particular q is compatible with some $\bar{p} \in A \cap N_{p_0}$. This is a contradiction since A is an antichain and $\bar{p} \neq p_1$. $\square$

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Definition

Given a cardinal κ, $\mathrm{MA}^{1.5}_{\kappa}$ is the following statement: Suppose $\mathbb{P}$ has the $\aleph_{1.5}$–c.c. and $\{D_i : i < \kappa\}$ is a collection of dense subsets of $\mathbb{P}$. Then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \kappa$.

Theorem

(A.–Mota) (CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–chain condition such that the following statements hold in the generic extension by $\mathcal{P}$:

(1) $2^{\aleph_0} = \kappa$

(2) $MA^{1.5}_{<2^{\aleph_0}}$

This is proved using a finite support iteration with partial symmetric systems of models as side conditions.

The extent of $\mathrm{MA}^{1.5}_\kappa \setminus \mathrm{MA}_\kappa$

Given an ordinal τ, we will say that a set X of ordinals is τ–*thin* in case $\mathrm{rank}(X, \delta) \leq \tau$ for all ordinals δ.

Definition

Given ordinals τ and λ, $\tau < \omega_1$, $(\cdot)^\tau_\lambda$ is the following statement:

For every sequence $(A_i)_{i<\lambda}$, if A_i is a τ–thin subset of ω_1 for all $i < \lambda$, then there is a club $C \subseteq \omega_1$ such that $|C \cap A_i| < \aleph_0$ for all i.

Fact

For every cardinal $\lambda \geq \omega_1$, $MA^{1.5}_\lambda$ implies $(\cdot)^\tau_\lambda$ for every $\tau < \omega_1$.

Proof.

Let $(A_i)_{i<\lambda}$ be as in the definition of $(\cdot)^\tau_\lambda$. Apply $\mathrm{MA}^{1.5}_\lambda$ to $\mathbb{P}$ consisting of all pairs (f, X) such that

(a) $f \subseteq \omega_1 \times \omega_1$ is a finite function such that $\mathrm{rank}(f(\nu), f(\nu)) \geq \nu$ for every $\nu \in \mathrm{dom}(f)$,

(b) X is finite set of triples (i, ν, a) such that $i < \lambda$, $\nu \in \mathrm{dom}(f)$, $\mathrm{rank}(A_i, f(\nu)) < f(\nu)$, and a is a finite subset of $f(\nu)$, and

(c) for every $(i, \nu, a) \in X$, $\mathrm{range}(f) \cap A_i = a$.

Given $\mathbb{P}$–conditions (f_0, X_0) and (f_1, X_1), (f_1, X_1) extends (f_0, X_0) if $f_0 \subseteq f_1$ and $X_0 \subseteq X_1$.

□

The $\mathrm{MA}^{1.5}_{\aleph_2}$ model is the first known model of $\mathrm{MA}_{\aleph_2}$ + ¬ WCG: In usual c.c.c. constructions of models of $\mathrm{MA}_{\aleph_2}$, once you add a Cohen real, you add a WCG–sequence $\vec{C}$. But then $\vec{C}$ remains WCG–sequence since the tail of the iteration is c.c.c.

Question

Does $MA^{1.5}_{\aleph_1}$ imply Measuring?

Question

Is $FA(\{\mathbb{P} : \mathbb{B}$ proper and with the $\aleph_2$–c.c.$\})_{\aleph_2}$ consistent?

Question

Is PFA restricted to proper forcings of cardinality $\aleph_1$ compatible with $2^{\aleph_0} > \aleph_2$?

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Does $MA^{1.5}_{\aleph_1}$ *imply Measuring?*

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Adding many Baumgartner clubs

Cohen's forcing $2^{<\omega}$ for adding a real is perhaps the simplest non-trivial forcing notion one can think of (and the first to be discovered).

A simple and nicely behaved forcing for adding any arbitrary number of Cohen reals: $\mathrm{Add}(\omega, X)$ = the partial order of finite functions $p : X \times 2^{<\omega}$, ordered by reverse inclusion.

For every $\mathrm{Add}(\omega, X)$–generic G and every $\alpha \in X$,

$$r_\alpha^G = \cup\{p(\alpha)\ :\ p \in G,\ ,\alpha \in \mathrm{dom}(p)\}$$

is a Cohen real over $\mathbf{V}$ and $r_\alpha^G \neq r_{\alpha'}^G$ for $\alpha \neq \alpha'$ in X.

Also:

- $\text{Add}(\omega, X)$ has the c.c.c.
- $\text{Add}(\omega, X)$ is homogeneous, in the sense that if p, $p' \in \text{Add}(\omega, X)$, then there are extensions q, q' of p, p', resp., such that $\text{Add}(\omega, X) \restriction q \cong \text{Add}(\omega, X) \restriction q'$.
- $\text{Add}(\omega, X)$ can be naturally represented as the product $\text{Add}(\omega, X_0) \times \text{Add}(\omega, X_1)$ for every partition (X_0, X_1) of X into nonempty pieces. In particular, for every $\text{Add}(\omega, X)$–generic G and all $\alpha \neq \alpha'$ in X, r_α^G is Cohen generic over $\mathbf{V}[r_{\alpha'}^G]$.

Cohen forcing and $\text{Add}(\omega, X)$ have of course been extensively studied for more than 50 years now. For example, $\text{Add}(\omega, X)$ is the forcing that Cohen used to prove the consistency of $\neg\,\text{CH}$ (by forcing over L).

A simple forcing for adding a new club of ω_1: Baumgartner's forcing $\mathbb{B}$ for adding a club of ω_1 with finite conditions:

$p \in \mathbb{B}$ iff $p \subseteq \omega_1 \times \omega_1$ is finite and can be extended to a strictly increasing and continuous function $f : \omega_1 \longrightarrow \omega_1$.
Extension: Reverse inclusion.

If G is $\mathbb{B}$–generic, then $\cup G$ is the enumerating function of a club C^G of ω_1 which does not contain any infinite set in the ground model.

- $|\mathbb{B}| = \aleph_1$
- $\mathbb{B}$ is proper: If N is any countable submodel of $H(\omega_2)$ and $p \in \mathbb{B} \cap N$, then $p \cup \{\langle N \cap \omega_1, N \cap \omega_1 \rangle\}$ is $(\mathbb{B}, N)$–generic.
- $\mathbb{B}$ is absolute between models agreeing on ω_1.

Also, Zapletal proved:

- If $x \in \mathbb{R}$, $x^\sharp$ exists, and $\mathcal{P} \in L[x]$ is a non–atomic partial order on $\omega_1^{\mathbf{V}}$, then $\mathcal{P}$ is forcing–equivalent to the disjoint sum of some number of copies of forcings in

$$\{2^{<\omega}, \mathrm{Add}(\omega, \omega_1), \mathbb{B}, \mathrm{Coll}(\omega, \omega_1)\}$$

- (PFA) $\mathbb{B}$ is a minimal nowhere c.c.c. poset (i.e., not c.c.c. below any condition) of size $\aleph_1$, in the sense that every nowhere c.c.c. poset of size $\aleph_1$ adds a generic for $\mathbb{B}$.
- If $P = \{p_\alpha : \alpha < \omega_1\}$ is a nowhere c.c.c. partial order adding a club $C \subseteq \omega_1$ such that for all $\alpha \in C$, $\dot{G} \cap \{p_\beta : \beta < \alpha\}$ is generic for $\{p_\beta : \beta < \alpha\}$, where $\dot{G}$ denotes the generic for P, then $\mathrm{RO}(P) = \mathrm{RO}(\mathbb{B})$.

Given a set X of ordinals, there is a forcing, which I will denote by $\mathrm{Add}_{\mathbb{B}}(X)$, which is *quite simple to define* and which has the following properties.

(1) For every $\mathrm{Add}_{\mathbb{B}}(X)$–generic G and every $\alpha \in X$ one can naturally extract a Baumgartner club C^G_α from G. Moreover, $C^G_\alpha \neq C^G_{\alpha'}$ for $\alpha \neq \alpha'$ in X.

(2) $\mathrm{Add}_{\mathbb{B}}(X)$ is proper and has the $\aleph_2$–c.c.

(3) $\mathrm{Add}_{\mathbb{B}}(X)$ is homogeneous (in the above sense).

(4) For every partition (X_0, X_1) of X into nonempty pieces, $\mathrm{Add}_{\mathbb{B}}(X) \cong \mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$. In particular, if G is $\mathrm{Add}_{\mathbb{B}}(X)$–generic and $\alpha \neq \alpha'$ are in X, then C^G_α is $\mathbb{B}$–generic over $\mathbf{V}[C^G_{\alpha'}]$.

Why is something non–trivial necessary?

Fact

Both the finite –support product of $\aleph_0$ copies of $\mathbb{B}$ and the full support produt of $\aleph_0$ copies of $\mathbb{B}$ collapse ω_1 (in fact, the full support product of $\aleph_0$ copies of Cohen forcing collapses ω_1).

Definintion: Let X be a set of ordinals. $\mathrm{Add}_{\mathbb{B}}(X)$ is the following forcing: Conditions in $\mathrm{Add}_{\mathbb{B}}(X)$ are pairs of the form $p = (f, \mathcal{S})$ with the following properties.

(1) f is a finite function with $\mathrm{dom}(f) \subseteq X$ and such that $f(\alpha) \in \mathbb{B}$ for every $\alpha \in \mathrm{dom}(f)$.

(2) $\mathcal{S}$ is a finite function with $\mathrm{dom}(\mathcal{S}) \subseteq \omega_1$ such that for every $\delta \in \mathrm{dom}(\mathcal{S})$,
   (a) δ is a countable indecomposable ordinal,
   (b) $\mathcal{S}(\delta)$ is a countable subset of X,
   (c) for all $\alpha \in \mathrm{dom}(f) \cap \mathcal{S}(\delta)$, $\delta \in \mathrm{dom}(f(\alpha))$ and $f(\alpha)(\delta) = \delta$, and
   (d) for every $\delta' \in \mathrm{dom}(\mathcal{S} \upharpoonright \delta)$ and every ordinal $\alpha \in \mathcal{S}(\delta)$, $\mathrm{rank}(\mathcal{S}(\delta'), \alpha) < \delta$.

Given $\mathrm{Add}_{\mathbb{B}}(X)$ conditions $(f_0, \mathcal{S}_0)$, $(f_1, \mathcal{S}_1)$, we say that $(f_1, \mathcal{S}_1)$ extends $(f_0, \mathcal{S}_0)$ iff

- $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$ and $f_0(\alpha) \subseteq f_1(\alpha)$ for every $\alpha \in \mathrm{dom}(f_0)$, and
- $\mathrm{dom}(\mathcal{S}_0) \subseteq \mathrm{dom}(\mathcal{S}_1)$ and $\mathcal{S}_0(\delta) \subseteq \mathcal{S}_1(\delta)$ for every $\delta \in \mathrm{dom}(\mathcal{S}_0)$.

The definition of $\mathrm{Add}_{\mathbb{B}}(X)$ is a streamlined version of previous constructions involving finite–support iterations with 'symmetric systems' of structures as side conditions. Given $\alpha \in X$ and an $\mathrm{Add}_{\mathbb{B}}(X)$–generic G, let

$$F_G^\alpha = \{f(\alpha) : (f, \mathcal{F}) \in G \text{ for some } \mathcal{F},\ \alpha \in \mathrm{dom}(f)\}$$

Fact

Given $\alpha \in X$ and an $\mathrm{Add}_{\mathbb{B}}(X)$–generic filter, F_G^α is $\mathbb{B}$–generic over $\mathbf{V}$.

Given a condition $p = (f, \mathcal{F}) \in \mathrm{Add}_{\mathbb{B}}(X)$ and $Y \subseteq X$, let

$$p \upharpoonright Y = (f \upharpoonright Y, \{\langle \delta, Z \cap Y \rangle : \langle \delta, Z \rangle \in \mathcal{F}\})$$

Also: Given functions $\mathcal{F}$ and $\mathcal{G}$, let $\mathcal{F} \oplus \mathcal{G}$ denote the function $\mathcal{H}$ with domain $\mathrm{dom}(\mathcal{F}) \cup \mathrm{dom}(\mathcal{G})$ such that

- $\mathcal{H}(x) = \mathcal{F}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \setminus \mathrm{dom}(\mathcal{G})$,
- $\mathcal{H}(x) = \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{G}) \setminus \mathrm{dom}(\mathcal{F})$, and
- $\mathcal{H}(x) = \mathcal{F}(x) \cup \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \cap \mathrm{dom}(\mathcal{G})$.

Lemma

Let X_0, X_1 be disjoint sets of ordinals. Then, the function sending $((f_0, \mathcal{F}_0), (f_1, \mathcal{F}_1)) \in \mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ to $(f_0 \cup f_1, \mathcal{F}_0 \oplus \mathcal{F}_1)$ is an isomorphism between $\mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ and $\mathrm{Add}_{\mathbb{B}}(X_0 \cup X_1)$. The inverse of this function is the function sending $p \in \mathrm{Add}_{\mathbb{B}}(X)$ to $(p \upharpoonright X_0, p \upharpoonright X_1)$.

Lemma

$\mathrm{Add}_{\mathbb{B}}(X)$ *has the* $\aleph_2$*–c.c.*

Proof using standard Δ–lemma (for collections of finite sets).
No need of CH.

Lemma

$\mathrm{Add}_{\mathbb{B}}(X)$ *is proper.*

Applications:

Proposition: If $\mathrm{ot}(X) \geq \omega_2$, then $\mathrm{Add}_{\mathbb{B}}(X)$ forces

- $\mathfrak{b}(\omega_1) = \aleph_2$
- $\mathfrak{d}(\omega_1) \geq |X|$

$\mathfrak{b}(\omega_1)$: Minimum κ s.t. there is $\mathcal{F} \subseteq {}^{\omega_1}\omega_1$ such that no $g : \omega_1 \longrightarrow \omega_1$ dominates each $f \in \mathcal{F}$ (mod. countable). And similarly one defines $\mathfrak{d}(\omega_1)$.

Proof: Identical to the corresponding proofs for $\mathfrak{b}$ and $\mathfrak{d}$ after forcing with $\mathrm{Add}(\omega, X)$ if $\mathrm{ot}(X) \geq \omega_1$. $\square$

Consider the following weak form of Club–Guessing at ω_1:

KA (Kunen's Axiom): There is a club–sequence $(C_\delta : \delta \in Lim(\omega_1))$ s.t. for every club $D \subseteq \omega_1$ there is some δ with $D \cap [C_\delta(n), C_\delta(n+1)) = \emptyset$ for co–finitely many $n < \omega$.

Definition Let $\mathcal{C} \subseteq \mathcal{P}(\omega_1)$ be such that $\mathrm{ot}(X) = \omega$ for all $X \in \mathcal{C}$. $\mathcal{C}$ is a KA–set if for every club $D \subseteq \omega_1$ there is some $X \in \mathcal{C}$ such that $D \cap [X(n), X(n+1)) \neq \emptyset$ for co–finitely many $n < \omega$.

Given a cardinal λ, KA_λ means: There is a KA–set of size at most λ.

Proposition A Baumgartner club destroys all KA–sets from the ground model. Hence, if $\mathrm{ot}(X) \geq \omega_2$, then $\mathrm{Add}_{\mathbb{B}}(X)$ forces $\neg\,\mathrm{KA}_\lambda$ for every $\lambda < |X|$.

Easy to get $\mathfrak{b}(\omega_1) = \aleph_2$ + $\mathfrak{d}(\omega_1) = 2^{\aleph_0} = 2^{\aleph_1}$ large by traditional means:

Add many Cohen subsets of ω_1 and then do c.c.c. forcing. But the first step forces $\diamondsuit$, and the second step preserves Club Guessing.

The consistency of $\neg\,\mathrm{KA}_\lambda$ for λ large was already known (it follows from $\mathrm{MA}^{1.5}_\lambda$). However, it is not clear how to get $\mathfrak{b}(\omega_1) = \aleph_2$ with these older constructions.

More on Club–Guessing

As we have seen, Club–Guessing, and related principles, at the level of ω_1 can be easily manipulated by forcing. On the other hand, instances of Club–Guessing at higher regular cardinals are often ZFC theorems.

Well–known example (Shelah): For every infinite regular cardinal κ and every stationary $S \subseteq S^{\kappa^{++}}_{\kappa} := \{\alpha < \kappa^{++} : \mathrm{cf}(\alpha) = \kappa\}$ there is a club–sequence $(C_\delta : \delta \in S)$ such that

- each C_δ is a club of δ of order type κ, and
- for every club $E \subseteq \kappa^{++}$ there is some $\delta \in S$ such that $C_\delta \subseteq E$.

In particular: There is always a club–guesssing sequence on $S^{\omega_2}_{\omega}$.

On the other hand: (Shelah) It is consistent that there is no club–guessing sequence on $S^{\omega_2}_{\omega_1}$. (Proof not straightforward.)

Question: Assume $\aleph_\omega$ is a strong limit. Is there a poset $\mathcal{P}$ with the following properties?

(a) $\mathcal{P}$ forces that there is a set $\mathcal{C}$ of subsets of ω_2 of order type ω_1 such that $|\mathcal{C}| = \aleph_2$ and such that for every club $D \subseteq \omega_2$ there is some $C \in \mathcal{C}$ such that $C \subseteq D$.

(b) $\mathcal{P}$ preserves ω_1, ω_2, and ω_3.

(c) $|\mathcal{P}| < \aleph_\omega$.

Why am I asking this?
If the answer is yes, then the following is a ZFC theorem: If $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_3}$.

Proof of this:

Suppose $2^{\aleph_\omega} > \aleph_{\omega_3}$. Force with $\mathcal{P}$ such that (a)–(c). In the extension, there is club–guessing set $\mathcal{C}$ as given by (a), $\aleph_\omega$ is strong limit, and $2^{\aleph_\omega} > \aleph_{\omega_3}$ still holds. Then (of course) run Shelah's proof for $2^{\aleph_\omega} < \aleph_{\omega_4}$ using $\mathcal{C}$ rather than a club–guessing sequence on $S^{\omega_3}_{\omega_1}$ and derive a contradiction in the same way.

Another ZFC theorem on club–guessing in the same region:

Notation: $X(\delta)$ is the δ–th member of X if X is a set of ordinals.

Theorem

(Shelah, Claim 3.3 in Colouring and non-productivity of $\aleph_2$–c.c., Ann. Pure and Applied Logic, vol. 84, 2 (1997), 153–174)

Let $\kappa > \omega_1$ be a regular cardinal. Then for every stationary $S \subseteq S^{\kappa^+}_\kappa$ there is a club–sequence $\langle C_\delta : \delta \in S\rangle$ such that for all $\delta \in S$,

- $\mathrm{ot}(C_\delta) = \kappa$, *and*
- $\mathrm{cf}(C_\delta(\alpha + 1)) = \kappa$ *for all* $\alpha < \kappa$,

and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in S$ (equivalently, stationary many $\delta \in S$) such that

$$\{\alpha < \kappa \ : \ C_\delta(\alpha + 1) \in D\}$$

is stationary.

See also D. Soukup and L. Soukup, *Club guessing for dummies* for a nicely written proof of the above.

Question (Shelah, Question 5.4 in *On what I do not understand (and have something to say): Part I*, Fundamenta Math., vol. 166, 1–2 (2000), 1–82.)

Is it true in ZFC that for every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\vec{C} = \langle C_\delta : \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ such that for every club $D \subseteq \kappa^+$ there is some δ such that

$$\{\alpha < \kappa : \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary?

According to Shelah in the above paper, if there is a club–sequence as in the above question on $S_{\kappa}^{\kappa^+}$ and GCH holds, then there is a κ^+–Souslin tree. In particular, an affirmative answer to above question would yield an affirmative answer to the following well–known open question.

Question: Does GCH imply that there is an ω_2–Souslin tree?

Theorem

(GCH) For every regular cardinal $\kappa \geq \omega_1$ there is a cardinal–preserving poset forcing that there is no club–sequence $\vec{C} = \langle C_\delta : \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is no δ such that

$$\{\alpha < \kappa : \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary.

Proof by a $<\kappa$–support iteration of length κ^{++} using, as side conditions, symmetric systems of size $<\kappa$ of models N such that $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$. Proof of relevant properness is not inductive.

References

(1) D. Asperó and M.A. Mota, *Forcing consequences of PFA together with the continuum large*, Transactions of the AMS, vol. 367 (2015), 6103–6129.

(2) D. Asperó and M.A. Mota, *A generalization of Martin's Axiom*, Israel J. of Mathematics, vol. 210 (2015), 193–231.

(3) D. Asperó, *The consistency of a club–guessing failure at the successor of a regular cardinal*, in "Infinity, computability, and metamathematics: Festschrift celebrating the 60th birthdays of Peter Koepke and Philip Welch," S. Geschke, B. Löwe and P. Schlicht, eds., College Publications, Tributes 23, London, 2014, 5–27.

(4) D. Asperó, *Adding many Baumgartner clubs*. To appear in a special volume dedicated to James Baumgartner.

(5) D. Asperó and M.A. Mota, *Separating club-guessing principles in the presence of fat forcing axioms*, Annals of Pure and Applied Logic, vol. 167, 3 (2016), 284–308.

(6) D. Asperó, *A forcing notion collapsing $\aleph_3$ and preserving all other cardinals*. Submitted (2016).

(7) D. Asperó and M.A. Mota, *Measuring club-sequences with the continuum large*. To appear in The Journal of Symbolic Logic (2017).