Lecture II
Recall from Lecture I

We are looking at ordinal \( \textbf{GLP} \)-spaces, i.e., polytopological spaces of the form \((\delta, (\tau_\zeta)_{\zeta<\xi})\), where \(\tau_0\) is the interval topology and \(\tau_{\zeta+1}\) is generated by \(\tau_\zeta\) together with the sets

\[
D_\zeta(A) := \{\alpha : \alpha \text{ is a } \tau_\zeta \text{ limit point of } A\}
\]

all \(A \subseteq \delta\).

\(\tau_1\) is the club topology. The non-isolated points are those \(\alpha\) with uncountable cofinality.

We observed that \(D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}\).
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We observed that \(D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}\).
Recall also the following definition

**Definition**

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in $\alpha$ (0-s-stationary in $\alpha$, for short) if and only if $A \cap \alpha$ is unbounded in $\alpha$.

For $\xi > 0$, we say that $A \subseteq \delta$ is $\xi$-simultaneously-stationary in $\alpha$ ($\xi$-s-stationary in $\alpha$, for short) if and only for every $\zeta < \xi$, every pair of $\zeta$-s-stationary subsets $B, C \subseteq \alpha$ simultaneously $\zeta$-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are $\zeta$-s-stationary in $\beta$.

$A$ is 2-s-stationary in $\alpha \iff$ every pair of stationary subsets of $\alpha$ simultaneously reflect to some $\beta \in A$. 
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$A$ is 2-s-stationary in $\alpha$ $\iff$ every pair of stationary subsets of $\alpha$ simultaneously reflect to some $\beta \in A$. 

An Introduction to Hyperstationary Sets
**Proposition**

\( \alpha \) is not isolated in the \( \tau_2 \) topology if and only if \( \alpha \) is 2-s-stationary

**Proof.**

If \( \alpha \) is not 2-s-stationary, there are stationary \( A, B \subseteq \alpha \) such that
\( D_1(A) \cap D_1(B) = \{ \alpha \} \), hence \( \alpha \) is isolated.

Now suppose \( \alpha \) is 2-s-stationary and \( \alpha \in U = C \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1}) \), where \( C \subseteq \alpha \) is club. We claim that \( U \) contains some ordinal other than \( \alpha \). It is enough to show that \( D_1(A_0) \cap \ldots \cap D_1(A_{n-1}) \) is stationary.

Suppose first that \( n = 2 \). Fix any club \( C' \subseteq \alpha \). The sets \( C' \cap A_0 \) and \( C' \cap A_1 \) are stationary in \( \alpha \), and therefore they simultaneously reflect at some \( \beta < \alpha \). Thus \( \beta \in C' \cap D_1(A_0) \cap D_1(A_1) \).

Now, assume it holds for \( n \) and let us show it holds for \( n + 1 \). Fix a club \( C' \subseteq \alpha \). By the ind. hyp., \( C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1}) \) is stationary. So, since the proposition holds for \( n = 2 \), the set
\( D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n) \) is also stationary. But clearly
\( D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n) \subseteq C' \cap D_1(A_0) \cap \ldots \cap D_1(A_n) \).
Proposition

$\alpha$ is not isolated in the $\tau_2$ topology if and only if $\alpha$ is 2-s-stationary

Proof.

If $\alpha$ is not 2-s-stationary, there are stationary $A, B \subseteq \alpha$ such that $D_1(A) \cap D_1(B) = \{\alpha\}$, hence $\alpha$ is isolated.

Now suppose $\alpha$ is 2-s-stat. and $\alpha \in U = C \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$, where $C \subseteq \alpha$ is club. We claim that $U$ contains some ordinal other than $\alpha$. It is enough to show that $D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary.

Suppose first that $n = 2$. Fix any club $C' \subseteq \alpha$. The sets $C' \cap A_0$ and $C' \cap A_1$ are stationary in $\alpha$, and therefore they simultaneously reflect at some $\beta < \alpha$. Thus $\beta \in C' \cap D_1(A_0) \cap D_1(A_1)$.

Now, assume it holds for $n$ and let us show it holds for $n+1$. Fix a club $C' \subseteq \alpha$. By the ind. hyp., $C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})$ is stationary. So, since the proposition holds for $n = 2$, the set $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n)$ is also stationary. But clearly $D_1(C' \cap D_1(A_0) \cap \ldots \cap D_1(A_{n-1})) \cap D_1(A_n) \subseteq C' \cap D_1(A_0) \cap \ldots \cap D_1(A_n)$.
A similar argument, relativized to any set $A$ yields:

**Proposition**

$$D_2(A) = \{\alpha : A \cap \alpha \text{ is 2-s-stationary in } \alpha\}.$$
The $\tau_\xi$ topology

In order to analyse the topologies $\tau_\xi$, for $\xi \geq 3$, note first the following general facts:

1. For every $\xi' < \xi$ and every $A, B \subseteq \delta$, 

$$D_{\xi'}(A) \cap D_\xi(B) = D_\xi(D_{\xi'}(A) \cap B).$$

2. For every ordinal $\xi$, the sets of the form 

$$I \cap D_{\xi'}(A_0) \cap \ldots \cap D_{\xi'}(A_{n-1})$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi' < \xi$, and $A_i \subseteq \delta$, all $i < n$, form a base for $\tau_\xi$. 
The $\tau_\xi$ topology

In order to analyse the topologies $\tau_\xi$, for $\xi \geq 3$, note first the following general facts:

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   where $I \in \mathcal{B}_0$, $n < \omega$, $\xi' < \xi$, and $A_i \subseteq \delta$, all $i < n$, form a base for $\tau_\xi$. 
Characterizing non-isolated points

**Theorem**

1. For every $\xi$,
   \[ D_\xi(A) = \{ \alpha : A \text{ is } \xi\text{-s-stationary in } \alpha \}. \]

2. For every $\xi$ and $\alpha$, $A$ is $\xi + 1$-s-stationary in $\alpha$ if and only if
   \[ A \cap D_\zeta(S) \cap D_\zeta(T) \cap \alpha \neq \emptyset \] (equivalently, if and only if
   \[ A \cap D_\zeta(S) \cap D_\zeta(T) \] is $\zeta$-s-stationary in $\alpha$) for every $\zeta \leq \xi$ and every
   pair $S, T$ of subsets of $\alpha$ that are $\zeta$-s-stationary in $\alpha$.

3. For every $\xi$ and $\alpha$, if $A$ is $\xi$-s-stationary in $\alpha$ and $A_i$ is $\zeta_i$-s-stationary
   in $\alpha$ for some $\zeta_i < \xi$, all $i < n$, then
   \[ A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1}) \]
   is $\xi$-s-stationary in $\alpha$.

\(^a\)For $\xi < \omega$, this is due independently to L. Beklemishev (Unpublished).
Characterizing non-isolated points

Theorem

1. For every $\xi$,

   $$D_\xi(A) = \{\alpha : A \text{ is } \xi\text{-s-stationary in } \alpha\}.$$ \(^a\)

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3. For every $\xi$ and $\alpha$, if $A$ is $\xi\text{-s-stationary in } \alpha$ and $A_i$ is $\zeta_i\text{-s-stationary}$

   in $\alpha$ for some $\zeta_i < \xi$, all $i < n$, then $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$

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Characterizing non-isolated points

**Theorem**

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3. For every $\xi$ and $\alpha$, if $A$ is $\xi$-s-stationary in $\alpha$ and $A_i$ is $\zeta_i$-s-stationary in $\alpha$ for some $\zeta_i < \xi$, all $i < n$, then $A \cap D_{\zeta_0}(A_0) \cap \ldots \cap D_{\zeta_{n-1}}(A_{n-1})$ is $\xi$-s-stationary in $\alpha$.

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Taking $A = \delta$ in (1) above, we obtain the following

**Corollary**

*For every $\xi$, an ordinal $\alpha < \delta$ is not isolated in the $\tau_\xi$ topology if and only if $\alpha$ is $\xi$-s-stationary.*
Taking $A = \delta$ in (1) above, we obtain the following

**Corollary**

For every $\xi$, an ordinal $\alpha < \delta$ is not isolated in the $\tau_\xi$ topology if and only if $\alpha$ is $\xi$-s-stationary.
The ideal of non-$\xi$-s-stationary sets

For each limit ordinal $\alpha$ and each $\xi$, let $NS^\xi_\alpha$ be the set of non-$\xi$-s-stationary subsets of $\alpha$.

Thus, if $\alpha$ has uncountable cofinality, $NS^1_\alpha$ is the ideal of non-stationary subsets of $\alpha$ and $(NS^1_\alpha)^*$ is the club filter over $\alpha$.

Notice that $\zeta \leq \xi$ implies $NS^\zeta_\alpha \subseteq NS^\xi_\alpha$ and $(NS^\zeta_\alpha)^* \subseteq (NS^\xi_\alpha)^*$.

Also note that $A \subseteq \alpha$ belongs to $\bigcap NS^\xi_\alpha$ if and only if for some $\zeta < \xi$ and some $\zeta$-s-stationary sets $S, T \subseteq \alpha$, the set $D_\zeta(S) \cap D_\zeta(T) \cap \alpha$ is contained in $A$. In particular, if $S \subseteq \alpha$ is $\zeta$-s-stationary, with $\zeta < \xi$, then $D_\zeta(S) \cap \alpha \in (NS^\xi_\alpha)^*$.
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For each limit ordinal $\alpha$ and each $\xi$, let $\text{NS}_{\alpha}^{\xi}$ be the set of non-$\xi$-s-stationary subsets of $\alpha$.

Thus, if $\alpha$ has uncountable cofinality, $\text{NS}_{\alpha}^{1}$ is the ideal of non-stationary subsets of $\alpha$ and $(\text{NS}_{\alpha}^{1})^{*}$ is the club filter over $\alpha$.

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Also note that $A \subseteq \alpha$ belongs to $(\text{NS}_{\alpha}^{\xi})^{*}$ if and only if for some $\zeta < \xi$ and some $\zeta$-s-stationary sets $S$, $T \subseteq \alpha$, the set $D_{\zeta}(S) \cap D_{\zeta}(T) \cap \alpha$ is contained in $A$. In particular, if $S \subseteq \alpha$ is $\zeta$-s-stationary, with $\zeta < \xi$, then $D_{\zeta}(S) \cap \alpha \in (\text{NS}_{\alpha}^{\xi})^{*}$. 
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Theorem

For every $\xi$, a limit ordinal $\alpha$ is $\xi$-s-stationary if and only if $\text{NS}_{\alpha}^{\xi}$ is a proper ideal, hence if and only if $(\text{NS}_{\alpha}^{\xi})^*$ is a proper filter.
Proof.

Assume $\alpha$ is $\xi$-s-stationary (hence $\alpha \not\in NS^\xi_\alpha$) and let us show that $NS^\xi_\alpha$ is an ideal. For $\xi = 0$ this is clear. So, suppose $\xi > 0$ and $A, B \in NS^\xi_\alpha$.

There exist $\zeta_A, \zeta_B < \xi$, and there exist sets $S_A, T_A \subseteq \alpha$ that are $\zeta_A$-s-stationary in $\alpha$, and sets $S_B, T_B \subseteq \alpha$ that are $\zeta_B$-s-stationary in $\alpha$, such that $D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap A = D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B) \cap B = \emptyset$. Hence,

$$(D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)) \cap (A \cup B) = \emptyset.$$

The set $X := D_{\zeta_A}(S_A) \cap D_{\zeta_A}(T_A) \cap D_{\zeta_B}(S_B) \cap D_{\zeta_B}(T_B)$ is $\max\{\zeta_A, \zeta_B\}$-s-stationary in $\alpha$. Now notice that

$$D_{\max\{\zeta_A, \zeta_B\}}(X) \subseteq X$$

and so we have

$$D_{\max\{\zeta_A, \zeta_B\}}(X) \cap \alpha \cap (A \cup B) = \emptyset$$

which witnesses that $A \cup B \in NS^\xi_\alpha$. 

\qed
Continued.

For the converse, assume $NS_\alpha^\xi$ is a proper ideal. Take any $A$ and $B$ $\zeta$-s-stationary subsets of $\alpha$, for some $\zeta < \xi$. Then $D_\zeta(A) \cap \alpha$ and $D_\zeta(B) \cap \alpha$ are in $(NS_\alpha^\xi)^*$. Moreover, if $S, T \subseteq \alpha$ are any $\zeta'$-s-stationary sets, for some $\zeta' < \xi$, then also $D_{\zeta'}(S) \cap \alpha$ and $D_{\zeta'}(T) \cap \alpha$ belong to $(NS_\alpha^\xi)^*$. Hence, since $(NS_\alpha^\xi)^*$ is a filter,

$$D_\zeta(A) \cap D_\zeta(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \in (NS_\alpha^\xi)^*$$

which implies, since $(NS_\alpha^\xi)^*$ is proper, that $D_\zeta(A) \cap D_\zeta(B) \cap D_{\zeta'}(S) \cap D_{\zeta'}(T) \cap \alpha \neq \emptyset$. This shows that $D_\zeta(A) \cap D_\zeta(B)$ is $\xi$-s-stationary in $\alpha$. Since $A$ and $B$ were arbitrary, this implies $\alpha$ is $\xi$-s-stationary.
Summary

The following are equivalent for every limit ordinal $\alpha$ and every $\xi > 0$:

1. $\alpha$ is a non-isolated point in the $\tau_\xi$ topology.
2. $\alpha$ is $\xi$-s-stationary.
3. $\mathcal{NS}_\alpha^\xi$ is a proper ideal.
Summary

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1. $\alpha$ is a non-isolated point in the $\tau_\xi$ topology.
2. $\alpha$ is $\xi$-s-stationary.
3. $NS^{\xi}_\alpha$ is a proper ideal.
Indescribable cardinals

Recall that a formula of second-order logic is $\Sigma^1_0$ (or $\Pi^1_0$) if it does not have quantifiers of second order, but it may have any finite number of first-order quantifiers and free first-order and second-order variables.

**Definition**

For $\xi$ any ordinal, we say that a formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is $\Pi^1_\xi$.

And a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is $\Sigma^1_\xi$. 
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$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is $\Sigma^1_\xi$. 
Definition

If $\xi$ is a limit ordinal, then we say that a formula is $\Pi^1_\xi$ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta$$

where $\varphi_\zeta$ is $\Pi^1_\zeta$, all $\zeta < \xi$, and it has only finitely-many free second-order variables. And we say that a formula is $\Sigma^1_\xi$ if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta$$

where $\varphi_\zeta$ is $\Sigma^1_\zeta$, all $\zeta < \xi$, and it has only finitely-many free second-order variables.
Definition

A cardinal $\kappa$ is $\Pi^1_\xi$-indescribable if for all subsets $A \subseteq V_\kappa$ and every $\Pi^1_\xi$ sentence $\varphi$, if

$$\langle V_\kappa, \in, A \rangle \models \varphi$$

then there is some $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi.$$
Theorem

Every $\Pi^1_\xi$-indescribable cardinal is $(\xi + 1)$-s-stationary. Hence, if $\xi$ is a limit ordinal and a cardinal $\kappa$ is $\Pi^1_\xi$-indescribable for all $\zeta < \xi$, then $\kappa$ is $\xi$-s-stationary.
Proof.

Let $\kappa$ be an infinite cardinal. Clearly, the fact that a set $A \subseteq \kappa$ is 0-s-stationary (i.e., unbounded) in $\kappa$ can be expressed as a $\Pi^1_0$ sentence $\varphi_0(A)$ over $\langle V_\kappa, \in, A \rangle$. Inductively, for every $\xi > 0$, the fact that a set $A \subseteq \kappa$ is $\xi$-s-stationary in $\kappa$ can be expressed by a $\Pi^1_\xi$ sentence $\varphi_\xi$ over $\langle V_\kappa, \in, A \rangle$. Namely,

$$\bigwedge_{\zeta < \xi} (A \text{ is } \zeta\text{-s-stationary})$$

in the case $\xi$ is a limit ordinal, and by the sentence

$$\bigwedge_{\zeta < \xi - 1} (A \text{ is } \zeta\text{-s-stationary}) \land$$

$$\forall S, T (S, T \text{ are } (\xi - 1)\text{-s-stationary in } \kappa \rightarrow$$

$$\exists \beta \in A (S \text{ and } T \text{ are } (\xi - 1)\text{-s-stationary in } \beta))$$

which is easily seen to be equivalent to a $\Pi^1_\xi$ sentence, in the case $\xi$ is a successor ordinal.
Continued.

Now suppose $\kappa$ is $\Pi^1_\xi$-indescribable, and suppose that $A$ and $B$ are $\zeta$-s-stationary subsets of $\kappa$, for some $\zeta \leq \xi$. Thus,

$$\langle V_\kappa, \in, A, B \rangle \models \varphi_\zeta[A] \land \varphi_\zeta[B].$$

By the $\Pi^1_\xi$-indescribability of $\kappa$ there exists $\beta < \kappa$ such that

$$\langle V_\beta, \in, A \cap \beta, B \cap \beta \rangle \models \varphi_\zeta[A \cap \beta] \land \varphi_\zeta[B \cap \beta]$$

which implies that $A$ and $B$ are $\zeta$-s-stationary in $\beta$. Hence $\kappa$ is $(\xi + 1)$-s-reflecting.
Reflection and indescribability in $L$


Assume $V = L$. For every $\xi > 0$, a regular cardinal is $(\xi + 1)$-stationary if and only if it is $\Pi^1_\xi$-indescribable, hence if and only if it is $(\xi + 1)$-s-stationary.$^{a,b}$

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The proof actually shows the following:

**Theorem**

Assume $V = L$. Suppose $\xi > 0$ and $\kappa$ is a regular $(\xi + 1)$-stationary cardinal. Then for every $A \subseteq \kappa$ and every $\Pi^1_\xi$ sentence $\Psi$, if $\langle L_\kappa, \in, A \rangle \models \Psi$, then there exists a $\xi$-stationary $S \subseteq \kappa$ such that $\Psi$ reflects to every ordinal $\lambda$ on which $S$ is $\xi$-stationary.
**Theorem**

\[ \text{CON}(\exists \kappa < \lambda \ (\kappa \text{ is } \Pi^1_\xi \text{-indescribable } \land \lambda \text{ is inaccessible})) \text{ implies } \text{CON}(\tau_{\xi+1} \text{ is non-discrete } \land \tau_{\xi+2} \text{ is discrete}). \]

**Proof.**

Let \( \kappa \) be \( \Pi^1_\xi \text{-indescribable} \), and let \( \lambda > \kappa \) be inaccessible. In \( L \), \( \kappa \) is \( \Pi^1_\xi \text{-indescribable} \) and \( \lambda \) is inaccessible. So, in \( L \), let \( \kappa_0 \) be the least \( \Pi^1_\xi \text{-indescribable} \) cardinal, and let \( \lambda_0 \) be the least inaccessible cardinal above \( \kappa_0 \). Then \( L_{\lambda_0} \) is a model of ZFC in which \( \kappa_0 \) is \( \Pi^1_\xi \text{-indescribable} \) and no regular cardinal greater than \( \kappa_0 \) is 2-stationary. The reason is that if \( \alpha \) is a regular cardinal greater than \( \kappa_0 \), then \( \alpha = \beta^+ \), for some cardinal \( \beta \). And since Jensen’s principle \( \Box_\beta \) holds, there exists a stationary subset of \( \alpha \) that does not reflect.
Theorem

\( \text{CON}(\exists \kappa < \lambda (\kappa \text{ is } \Pi^1_\xi \text{-indescribable } \land \lambda \text{ is inaccessible})) \) implies \( \text{CON}(\tau_{\xi+1} \text{ is non-discrete } \land \tau_{\xi+2} \text{ is discrete}) \).

Proof.

Let \( \kappa \) be \( \Pi^1_\xi \)-indescribable, and let \( \lambda > \kappa \) be inaccessible. In \( L \), \( \kappa \) is \( \Pi^1_\xi \)-indescribable and \( \lambda \) is inaccessible. So, in \( L \), let \( \kappa_0 \) be the least \( \Pi^1_\xi \)-indescribable cardinal, and let \( \lambda_0 \) be the least inaccessible cardinal above \( \kappa_0 \). Then \( L_{\lambda_0} \) is a model of ZFC in which \( \kappa_0 \) is \( \Pi^1_\xi \)-indescribable and no regular cardinal greater than \( \kappa_0 \) is 2-stationary. The reason is that if \( \alpha \) is a regular cardinal greater than \( \kappa_0 \), then \( \alpha = \beta^+ \), for some cardinal \( \beta \). And since Jensen’s principle \( \square_\beta \) holds, there exists a stationary subset of \( \alpha \) that does not reflect.
On the consistency strength of 2-stationarity

Let us write:

\[ d_\xi(A) := \{ \alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha \} \]

Definition (A. H. Mekler-S. Shelah, 1989)

A regular uncountable cardinal \( \kappa \) is a reflection cardinal if there exists a proper, normal, and \( \kappa \)-complete ideal \( \mathcal{I} \) on \( \kappa \) such that for every \( X \subseteq \kappa \),

\[ X \in \mathcal{I}^+ \implies d_1(X) \in \mathcal{I}^+. \]

Note: if \( \kappa \) is 2-stationary, then \( NS_\kappa \) is the smallest such ideal.

\( \kappa \) is weakly compact \( \implies \) many reflection cardinals below \( \kappa \).
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The consistency strength of hyperstationarity. Applications and Open Questions

On the consistency strength of 2-stationarity

**Theorem (A. H. Mekler-S. Shelah, 1989)**

If \( \kappa \) is a reflection cardinal in \( L \), then in some generic extension of \( L \) that preserves cardinals, \( \kappa \) is 2-stationary. (In fact, the set \( \text{Reg} \cap \kappa \) of regular cardinals below \( \kappa \) is 2-stationary).

**Corollary**

The following are equiconsistent:

1. There exists a reflection cardinal.
2. There exists a 2-stationary cardinal.
3. There exists a regular cardinal \( \kappa \) such that every \( \kappa \)-free abelian group is \( \kappa^+ \)-free.
On the consistency strength of 2-stationarity

**Theorem (A. H. Mekler-S. Shelah, 1989)**

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On the consistency strength of 2-stationarity

Recall that a regular cardinal \( \kappa \) is **greatly Mahlo** if there exists a proper, normal, and \( \kappa \)-complete ideal \( \mathcal{I} \) on \( \kappa \) such that \( \text{Reg} \cap \kappa \in \mathcal{I}^* \), and for every \( X \subseteq \kappa \),

\[
X \in \mathcal{I}^* \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^*.
\]

**Theorem (A. H. Mekler-S. Shelah, 1989)**

*In L, if \( \kappa \) is at most the first greatly-Mahlo cardinal, then \( \kappa \) is not a reflection cardinal.*

Thus, in \( L \), the first reflection cardinal is strictly between the first greatly-Mahlo and the first weakly-compact.
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Recall that a regular cardinal $\kappa$ is **greatly Mahlo** if there exists a proper, normal, and $\kappa$-complete ideal $\mathcal{I}$ on $\kappa$ such that $\text{Reg} \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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On the consistency strength of $n$-stationarity

We would like to prove analogous results for the $n$-stationary sets. So, let’s define:

**Definition**

For $n > 0$, a regular uncountable cardinal $\kappa$ is an $n$-reflection cardinal if there exists a proper, normal, and $\kappa$-complete ideal $\mathcal{I}$ on $\kappa$ such that for every $X \subseteq \kappa$,

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**Note:** If $\kappa$ is $n$-s-stationary, then the set $\text{NS}_{\kappa}^n$ of non-$n$-s-stationary subsets of $\kappa$ is the least such ideal.
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On the consistency strength of $n$-stationarity

**Theorem (J.B., M. Magidor, and S. Mancilla, 2015)**

*If $\kappa$ is a 2-reflection cardinal in $L$, then in some generic extension of $L$ that preserves cardinals, $\kappa$ is 3-stationary.*

*(In fact, the set $\text{Reg} \cap \kappa$ of regular cardinals below $\kappa$ is 3-stationary).*

Similar arguments should yield a similar result for $n > 3$. 
On the consistency strength of $n$-stationarity

Theorem (J.B., M. Magidor, and S. Mancilla, 2015)

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On the consistency strength of $n$-stationarity

**Definition**

A regular cardinal $\kappa$ is **$n$-greatly Mahlo** if there exists a proper, normal, and $\kappa$-complete ideal $\mathcal{I}$ on $\kappa$ such that $\text{Reg} \cap \kappa \in \mathcal{I}^*$, and for every $X \subseteq \kappa$,

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**Theorem (J.B. and S. Mancilla, 2014)**

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On the consistency strength of $n$-s-stationarity.

Magidor\(^1\) shows that the following are equiconsistent:

1. There exists a 2-s-stationary cardinal (i.e., a cardinal that reflects simultaneously pairs of stationary sets).
2. There exists a weakly-compact cardinal.

**Conjecture**

The following should be equiconsistent for every $n > 0$:

1. There exists an $(n + 1)$-s-stationary cardinal.
2. There exists an $\Pi^1_n$-indescribable cardinal.

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\(^1\)M. Magidor, On reflecting stationary sets. JSL 47 (1982)
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