An Introduction to Hyperstationary Sets

Joan Bagaria

Winter School in Abstract Analysis 2017
section Set Theory & Topology
Hejnice, Czech Republic, Jan 28 - Feb 4, 2017

Outline

1. Introduction: derived topologies and hyperstationary sets
2. Hyperstationary sets and indescribable cardinals
3. The consistency strength of hyperstationarity. Applications and Open Questions

Provability Logic

Provability Logic is the logic in the language of propositional logic with an additional modal operator $\Box$.

Axioms:

1. Boolean tautologies.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash \Box\varphi$ (Necessitation)

Provability Logic

Provability Logic is the logic in the language of propositional logic with an additional modal operator $\Box$.

Axioms:

1. Boolean tautologies.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash \Box\varphi$ (Necessitation)

Provability Logic

Provability Logic is the logic in the language of propositional logic with an additional modal operator $\Box$.

Axioms:

1. Boolean tautologies.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash \Box\varphi$ (Necessitation)

The Logic $\mathbf{GLP}_\omega$

One may introduce additional modal operators $[n]$, for each $n < \omega$. The corresponding dual operators $\neg[n]\neg$ are denoted by $\langle n \rangle$. The logic system $\mathbf{GLP}_\omega$ (Japaridze, 1986) has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$, for all $n < \omega$.
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$, for all $n < \omega$.
4. $[m]\varphi \to [n]\varphi$, for all $m < n < \omega$.
5. $\langle m \rangle \varphi \to [n]\langle m \rangle \varphi$, for all $m < n < \omega$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [n]\varphi$, for all $n < \omega$ (Necessitation)

The Logic $\mathbf{GLP}_\omega$

One may introduce additional modal operators $[n]$, for each $n < \omega$. The corresponding dual operators $\neg[n]\neg$ are denoted by $\langle n\rangle$. The logic system $\mathbf{GLP}_\omega$ (Japaridze, 1986) has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$, for all $n < \omega$.
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$, for all $n < \omega$.
4. $[m]\varphi \to [n]\varphi$, for all $m < n < \omega$.
5. $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for all $m < n < \omega$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [n]\varphi$, for all $n < \omega$ (Necessitation)

The Logic $\mathbf{GLP}_\omega$

One may introduce additional modal operators $[n]$, for each $n < \omega$. The corresponding dual operators $\neg[n]\neg$ are denoted by $\langle n \rangle$. The logic system $\mathbf{GLP}_\omega$ (Japaridze, 1986) has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$, for all $n < \omega$.
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$, for all $n < \omega$.
4. $[m]\varphi \to [n]\varphi$, for all $m < n < \omega$.
5. $\langle m \rangle \varphi \to [n]\langle m \rangle \varphi$, for all $m < n < \omega$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [n]\varphi$, for all $n < \omega$ (Necessitation)

The Logic $\mathbf{GLP}_\xi$

More generally, for any ordinal $\xi \geq 2$, one considers the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The corresponding dual operators $\neg[\alpha]\neg$ being denoted by $\langle\alpha\rangle$. The logic system $\mathbf{GLP}_\xi$ has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, for all $\alpha < \xi$.
3. $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$, for all $\alpha < \xi$.
4. $[\beta]\varphi \to [\alpha]\varphi$, for all $\beta < \alpha < \xi$.
5. $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for all $\beta < \alpha < \xi$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [\alpha]\varphi$, for all $\alpha < \xi$ (Necessitation)

The Logic $\mathbf{GLP}_\xi$

More generally, for any ordinal $\xi \geq 2$, one considers the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The corresponding dual operators $\neg[\alpha]\neg$ being denoted by $\langle\alpha\rangle$. The logic system $\mathbf{GLP}_\xi$ has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, for all $\alpha < \xi$.
3. $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$, for all $\alpha < \xi$.
4. $[\beta]\varphi \to [\alpha]\varphi$, for all $\beta < \alpha < \xi$.
5. $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for all $\beta < \alpha < \xi$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [\alpha]\varphi$, for all $\alpha < \xi$ (Necessitation)

The Logic $\mathbf{GLP}_\xi$

More generally, for any ordinal $\xi \geq 2$, one considers the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The corresponding dual operators $\neg[\alpha]\neg$ being denoted by $\langle\alpha\rangle$. The logic system $\mathbf{GLP}_\xi$ has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, for all $\alpha < \xi$.
3. $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$, for all $\alpha < \xi$.
4. $[\beta]\varphi \to [\alpha]\varphi$, for all $\beta < \alpha < \xi$.
5. $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for all $\beta < \alpha < \xi$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [\alpha]\varphi$, for all $\alpha < \xi$ (Necessitation)

Topological semantics

People have been interested in proving completeness for $\mathbf{GLP}_\xi$, with respect to some natural semantics.

Problem: Kripke-style semantics do not work!

So the goal has been to prove completeness for $\mathbf{GLP}_\xi$ with respect to topological semantics.

Topological semantics

People have been interested in proving completeness for $\mathbf{GLP}_\xi$, with respect to some natural semantics.

Problem: Kripke-style semantics do not work!

So the goal has been to prove completeness for $\mathbf{GLP}_\xi$ with respect to topological semantics.

Topological semantics

People have been interested in proving completeness for $\mathbf{GLP}_\xi$, with respect to some natural semantics.

Problem: Kripke-style semantics do not work!

So the goal has been to prove completeness for $\mathbf{GLP}_\xi$ with respect to topological semantics.

Topological semantics

Thus, one considers polytopological spaces $(X, (\tau_\alpha)_{\alpha<\xi})$.

A valuation on X is a map $v : Form \to \mathcal{P}(X)$ such that:

1. $v(\neg\varphi) = X - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle\alpha\rangle\varphi) = D_\alpha(v(\varphi))$, for all $\alpha < \xi$, where $D_\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator for τ_α (i.e., $D_\alpha(A)$ is the set of limit points of A in the τ_α topology).
   Hence, $v([\alpha]\varphi) = X - D_\alpha(X - v(\varphi))$ = the τ_α-interior of $v(\varphi)$, for all $\alpha < \xi$.

A formula is valid in X if $v(\varphi) = X$, for every valuation v on X.

Topological semantics

Thus, one considers polytopological spaces $(X, (\tau_\alpha)_{\alpha<\xi})$.

A valuation on X is a map $v : Form \to \mathcal{P}(X)$ such that:

1. $v(\neg\varphi) = X - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle\alpha\rangle\varphi) = D_\alpha(v(\varphi))$, for all $\alpha < \xi$, where $D_\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator for τ_α (i.e., $D_\alpha(A)$ is the set of limit points of A in the τ_α topology).
   Hence, $v([\alpha]\varphi) = X - D_\alpha(X - v(\varphi)) =$ the τ_α-interior of $v(\varphi)$, for all $\alpha < \xi$.

A formula is valid in X if $v(\varphi) = X$, for every valuation v on X.

Topological semantics

Thus, one considers polytopological spaces $(X, (\tau_\alpha)_{\alpha<\xi})$.

A valuation on X is a map $v : Form \to \mathcal{P}(X)$ such that:

1. $v(\neg\varphi) = X - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle\alpha\rangle\varphi) = D_\alpha(v(\varphi))$, for all $\alpha < \xi$, where $D_\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator for τ_α (i.e., $D_\alpha(A)$ is the set of limit points of A in the τ_α topology).
   Hence, $v([\alpha]\varphi) = X - D_\alpha(X - v(\varphi)) =$ the τ_α-interior of $v(\varphi)$, for all $\alpha < \xi$.

A formula is valid in X if $v(\varphi) = X$, for every valuation v on X.

Topological semantics

For the $\mathbf{GLP}_\xi$ axioms to be valid in $(X, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. $D_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

Moreover, for $\mathbf{GLP}_\xi$ to be complete, one must also have:

4. The τ_α are non-trivial (i.e., non discrete).

So, one doesn't have much choice on how to define the τ_α: One fixes a scattered topology τ_0 on X, and the other topologies are determined by the D_α operators. One only needs to make sure the τ_α are non-trivial.

Such polytopological spaces are called general **GLP**-spaces.

Topological semantics

For the $\mathbf{GLP}_\xi$ axioms to be valid in $(X, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. $D_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

   Moreover, for $\mathbf{GLP}_\xi$ to be complete, one must also have:
4. The τ_α are non-trivial (i.e., non discrete).

So, one doesn't have much choice on how to define the τ_α: One fixes a scattered topology τ_0 on X, and the other topologies are determined by the D_α operators. One only needs to make sure the τ_α are non-trivial.

Such polytopological spaces are called general **GLP**-spaces.

Topological semantics

For the $\mathbf{GLP}_\xi$ axioms to be valid in $(X, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. $D_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

   Moreover, for $\mathbf{GLP}_\xi$ to be complete, one must also have:
4. The τ_α are non-trivial (i.e., non discrete).

So, one doesn't have much choice on how to define the τ_α: One fixes a scattered topology τ_0 on X, and the other topologies are determined by the D_α operators. One only needs to make sure the τ_α are non-trivial.

Such polytopological spaces are called general **GLP**-spaces.

Topological semantics

For the $\mathbf{GLP}_\xi$ axioms to be valid in $(X, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. $D_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

   Moreover, for $\mathbf{GLP}_\xi$ to be complete, one must also have:
4. The τ_α are non-trivial (i.e., non discrete).

So, one doesn't have much choice on how to define the τ_α: One fixes a scattered topology τ_0 on X, and the other topologies are determined by the D_α operators. One only needs to make sure the τ_α are non-trivial.

Such polytopological spaces are called general **GLP**-spaces.

Ordinal **GLP**-spaces

Fix some limit ordinal δ (we also allow $\delta = OR$).

Recall that the order topology on δ (a. k. a. the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are precisely the limit ordinals.

Now define a continuous sequence of derived topologies

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_\xi \subseteq \ldots$$

as follows:

Ordinal **GLP**-spaces

Fix some limit ordinal δ (we also allow $\delta = OR$).

Recall that the order topology on δ (a. k. a. the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are precisely the limit ordinals.

Now define a continuous sequence of derived topologies

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_\xi \subseteq \ldots$$

as follows:

Ordinal **GLP**-spaces

Fix some limit ordinal δ (we also allow $\delta = OR$).

Recall that the order topology on δ (a. k. a. the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are precisely the limit ordinals.

Now define a continuous sequence of derived topologies

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_\xi \subseteq \ldots$$

as follows:

Ordinal **GLP**-spaces

Fix some limit ordinal δ (we also allow $\delta = OR$).

Recall that the order topology on δ (a. k. a. the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are precisely the limit ordinals.

Now define a continuous sequence of derived topologies

$$\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_\xi \subseteq \ldots$$

as follows:

Derived Topologies

Given τ_ξ, let $D_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator:

$$D_\xi(A) := \{\alpha \in \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Note that $D_\xi(A)$ is a closed set in the τ_ξ topology.

Then let $\tau_{\xi+1}$ be the topology generated by the set

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{D_\xi(A) : A \subseteq \delta\}.$$

Derived Topologies

Given τ_ξ, let $D_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator:

$$D_\xi(A) := \{\alpha \in \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Note that $D_\xi(A)$ is a closed set in the τ_ξ topology.

Then let $\tau_{\xi+1}$ be the topology generated by the set

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{D_\xi(A) : A \subseteq \delta\}.$$

Derived Topologies

Given τ_ξ, let $D_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator:

$$D_\xi(A) := \{\alpha \in \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Note that $D_\xi(A)$ is a closed set in the τ_ξ topology.

Then let $\tau_{\xi+1}$ be the topology generated by the set

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{D_\xi(A) : A \subseteq \delta\}.$$

Derived Topologies

Notice that if the cofinality of α is uncountable and $\alpha \in D_0(A)$, then $D_0(A) \cap \alpha$ is a club subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{D_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on OR, known as the club topology.

Note that the non-isolated points are exactly the ordinals of uncountable cofinality.

Fact

For every set of ordinals A,

$$D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

Derived Topologies

Notice that if the cofinality of α is uncountable and $\alpha \in D_0(A)$, then $D_0(A) \cap \alpha$ is a club subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{D_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on OR, known as the club topology.

Note that the non-isolated points are exactly the ordinals of uncountable cofinality.

Fact

For every set of ordinals A,

$$D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

Derived Topologies

Notice that if the cofinality of α is uncountable and $\alpha \in D_0(A)$, then $D_0(A) \cap \alpha$ is a club subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{D_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on OR, known as the club topology.

Note that the non-isolated points are exactly the ordinals of uncountable cofinality.

Fact

For every set of ordinals A,

$$D_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

Derived Topologies

Notice that if the cofinality of α is uncountable and $\alpha \in D_0(A)$, then $D_0(A) \cap \alpha$ is a club subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{D_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on OR, known as the club topology.

Note that the non-isolated points are exactly the ordinals of uncountable cofinality.

Fact

For every set of ordinals A,

$$D_1(A) = \{\alpha : A \cap \alpha \textit{ is stationary in } \alpha\}.$$

Derived Topologies

The next topology, τ_2, is generated by the set

$$\mathcal{B}_2 := \mathcal{B}_1 \cup \{D_1(A) : A \subseteq OR\}.$$

If some stationary subset S of α does not reflect (i.e., $D_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point α has to reflect all stationary sets.

Further, if some stationary subsets S, T of α do not simultaneously reflect (i.e., $D_1(S) \cap D_1(T) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point has to reflect simultaneously all pairs of stationary sets.

Derived Topologies

The next topology, τ_2, is generated by the set

$$\mathcal{B}_2 := \mathcal{B}_1 \cup \{D_1(A) : A \subseteq OR\}.$$

If some stationary subset S of α does not reflect (i.e., $D_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point α has to reflect all stationary sets.

Further, if some stationary subsets S, T of α do not simultaneously reflect (i.e., $D_1(S) \cap D_1(T) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point has to reflect simultaneously all pairs of stationary sets.

Derived Topologies

The next topology, τ_2, is generated by the set

$$\mathcal{B}_2 := \mathcal{B}_1 \cup \{D_1(A) : A \subseteq OR\}.$$

If some stationary subset S of α does not reflect (i.e., $D_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point α has to reflect all stationary sets.

Further, if some stationary subsets S, T of α do not simultaneously reflect (i.e., $D_1(S) \cap D_1(T) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point has to reflect simultaneously all pairs of stationary sets.

Stationary reflection

An ordinal α of uncountable cofinality reflects stationary sets if for every stationary $A \subseteq \alpha$ there exists $\beta < \alpha$ such that $A \cap \beta$ is stationary in β.

Let us say that an ordinal α of uncountable cofinality is simultaneoulsy-stationary-reflecting if every pair A, B of stationary subsets of α simultaneously reflect, that is, there exists $\beta < \alpha$ such that $A \cap \beta$ and $B \cap \beta$ are both stationary in β.

Stationary reflection

An ordinal α of uncountable cofinality reflects stationary sets if for every stationary $A \subseteq \alpha$ there exists $\beta < \alpha$ such that $A \cap \beta$ is stationary in β.

Let us say that an ordinal α of uncountable cofinality is simultaneoulsy-stationary-reflecting if every pair A, B of stationary subsets of α simultaneously reflect, that is, there exists $\beta < \alpha$ such that $A \cap \beta$ and $B \cap \beta$ are both stationary in β.

Jensen's Theorem

It is easy to see that every weakly-compact cardinal (i.e., Π^1_1-indescribable) is simultaneously-stationary-reflecting.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ reflects stationary sets if and only if it is Π^1_1-indescribable, hence if and only if it is simultaneously-stationary-reflecting.[a]

[a] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

Thus, in L, the non-isolated points of the topology τ_2 are precisely the ordinals whose cofinality is a weakly-compact cardinal.

Jensen's Theorem

It is easy to see that every weakly-compact cardinal (i.e., Π^1_1-indescribable) is simultaneously-stationary-reflecting.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ reflects stationary sets if and only if it is Π^1_1-indescribable, hence if and only if it is simultaneously-stationary-reflecting.[a]

[a] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

Thus, in L, the non-isolated points of the topology τ_2 are precisely the ordinals whose cofinality is a weakly-compact cardinal.

Jensen's Theorem

It is easy to see that every weakly-compact cardinal (i.e., Π^1_1-indescribable) is simultaneously-stationary-reflecting.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ reflects stationary sets if and only if it is Π^1_1-indescribable, hence if and only if it is simultaneously-stationary-reflecting.[a]

[a] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

Thus, in L, the non-isolated points of the topology τ_2 are precisely the ordinals whose cofinality is a weakly-compact cardinal.

Moreover,

Theorem (Magidor)

If κ is regular and reflects simultaneously pairs of stationary subsets, then κ is a weakly compact cardinal in L.[a]

[a] M. Magidor, Reflecting stationary sets. JSL, Vol. 47, Num. 4 (1982)

It follows that the consistency strength of the non-triviality of τ_2 is a weakly compact cardinal.

Moreover,

Theorem (Magidor)

If κ is regular and reflects simultaneously pairs of stationary subsets, then κ is a weakly compact cardinal in L.[a]

[a] M. Magidor, Reflecting stationary sets. JSL, Vol. 47, Num. 4 (1982)

It follows that the consistency strength of the non-triviality of τ_2 is a weakly compact cardinal.

ξ-stationary sets

Definition

We say that $A \subseteq \delta$ is 0-stationary in α, α a limit ordinal, if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that A is ξ-stationary in α if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

Note:

1. A is 1-stationary in $\alpha \Leftrightarrow A$ is stationary in α, in the usual sense.
2. A is 2-stationary in $\alpha \Leftrightarrow$ every stationary subset of α reflects to some $\beta \in A$.

ξ-stationary sets

Definition

We say that $A \subseteq \delta$ is 0-stationary in α, α a limit ordinal, if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that A is ξ-stationary in α if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

Note:

1. A is 1-stationary in $\alpha \Leftrightarrow A$ is stationary in α, in the usual sense.
2. A is 2-stationary in $\alpha \Leftrightarrow$ every stationary subset of α reflects to some $\beta \in A$.

ξ-stationary sets

Definition

We say that $A \subseteq \delta$ is 0-stationary in α, α a limit ordinal, if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that A is ξ-stationary in α if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

Note:

1. A is 1-stationary in $\alpha \Leftrightarrow A$ is stationary in α, in the usual sense.
2. A is 2-stationary in $\alpha \Leftrightarrow$ every stationary subset of α reflects to some $\beta \in A$.

ξ-stationary reflection

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.

For $\xi > 0$, we say that $A \subseteq \delta$ is ξ-simultaneously-stationary in α (ξ-s-stationary in α, for short) if and only for every $\zeta < \xi$, every pair of ζ-s-stationary subsets $B, C \subseteq \alpha$ simultaneously ζ-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are ζ-s-stationary in β.

Note:

1. A is 1-s-stationary in $\alpha \Leftrightarrow A$ is stationary in α.
2. A is 2-s-stationary in $\alpha \Leftrightarrow$ every pair of stationary subsets of α simultaneously reflect to some $\beta \in A$.

ξ-stationary reflection

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that $A \subseteq \delta$ is ξ-simultaneously-stationary in α (ξ-s-stationary in α, for short) if and only for every $\zeta < \xi$, every pair of ζ-s-stationary subsets $B, C \subseteq \alpha$ simultaneously ζ-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are ζ-s-stationary in β.

Note:

1. A is 1-s-stationary in $\alpha \Leftrightarrow A$ is stationary in α.
2. A is 2-s-stationary in $\alpha \Leftrightarrow$ every pair of stationary subsets of α simultaneously reflect to some $\beta \in A$.

ξ-stationary reflection

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that $A \subseteq \delta$ is ξ-simultaneously-stationary in α (ξ-s-stationary in α, for short) if and only for every $\zeta < \xi$, every pair of ζ-s-stationary subsets $B, C \subseteq \alpha$ simultaneously ζ-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are ζ-s-stationary in β.

Note:

1. A is 1-s-stationary in $\alpha \Leftrightarrow A$ is stationary in α.
2. A is 2-s-stationary in $\alpha \Leftrightarrow$ every pair of stationary subsets of α simultaneously reflect to some $\beta \in A$.

ξ-stationary reflection

Definition

We say that $A \subseteq \delta$ is 0-simultaneously-stationary in α (0-s-stationary in α, for short) if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that $A \subseteq \delta$ is ξ-simultaneously-stationary in α (ξ-s-stationary in α, for short) if and only for every $\zeta < \xi$, every pair of ζ-s-stationary subsets $B, C \subseteq \alpha$ simultaneously ζ-s-reflect at some $\beta \in A$, i.e., $B \cap \beta$ and $C \cap \beta$ are ζ-s-stationary in β.

Note:

1. A is 1-s-stationary in $\alpha \Leftrightarrow A$ is stationary in α.
2. A is 2-s-stationary in $\alpha \Leftrightarrow$ every pair of stationary subsets of α simultaneously reflect to some $\beta \in A$.

Lecture II