Generalizing Schreier families to large index sets III

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From previous lectures

- Given $\mathcal{F}$ on κ and $\mathcal{H}$ on ω, $\mathcal{G}$ on κ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ if every infinite sequence $(s_n)_n$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that, for every $x \in \mathcal{H}$, $\bigcup_{n \in x} t_n \in \mathcal{G}$.

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- For any tree T, we consider the partial orders $<_a$ and $<_c$ on T, whose chains are sets of immediate successors of a single node and usual chains, respectively.

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- For any tree T, we consider the partial orders $<_a$ and $<_c$ on T, whose chains are sets of immediate successors of a single node and usual chains, respectively.
- Given $\mathcal{A}$ and $\mathcal{C}$ on T, $\mathcal{A} \odot \mathcal{C}$ is the family of finite subsets s of T such that:
    - the chains of $\langle s \rangle$ with respect to $<_c$ belong to $\mathcal{C}$ (as in the case of the binary tree);
    - and for every $t \in T$, the set of immediate successors of t below some element of $\langle s \rangle$ belongs to $\mathcal{A}$.

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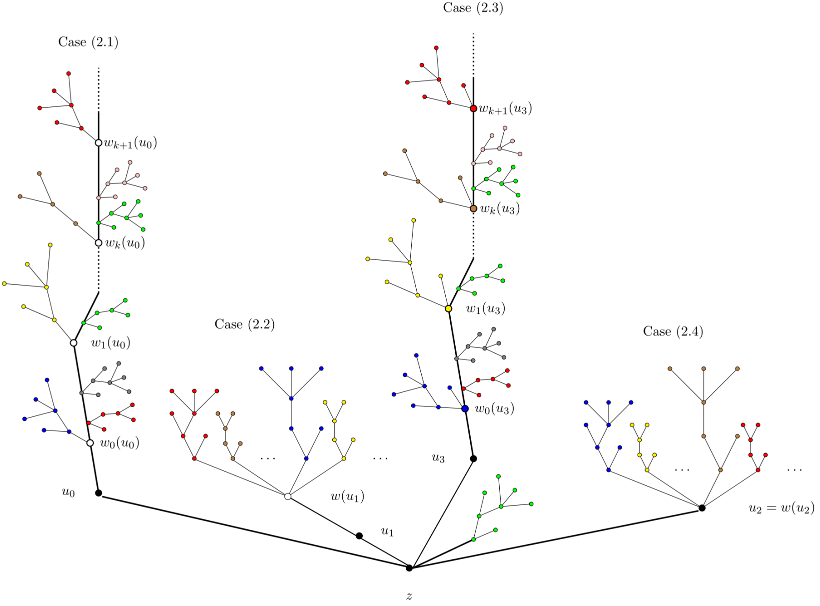
Theorem 21

If $\mathcal{A}$ and $\mathcal{C}$ are hereditary and compact, then so is $\mathcal{A} \odot \mathcal{C}$.

If $(\tau_k)_k$ is a sequence of finite subtrees of T, there is a subsequence $(\tau_k)_{k\in M}$ which is a Δ-system and ...



Combinatorial analysis

The following is a consequence of the combinatorial lemma relative to the picture (Lemma 22).

Theorem 23

If $\mathcal{A}_1$ and $\mathcal{C}_1$ are a multiplication of $\mathcal{A}_0$ and $\mathcal{C}_0$ by $\mathcal{S}$ respectively, then $(\mathcal{A}_1 \sqcup_a [T]^{\leq 1}) \odot (\mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1)$ is a multiplication of $\mathcal{A}_0 \odot \mathcal{C}_0$ by $\mathcal{S}$

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Corollary 24

If there are CL-sequences on chains of $(T, <_c)$ and of $(T, <_a)$, then there is a CL-sequence on T (with any total order).

First main result

Theorem 25 (Todorcevic)

For every strongly inaccessible cardinal κ, κ is Mahlo cardinal iff there is no special κ-Aronszajn tree, ie. a tree $(T, <)$ of height κ with no cofinal branches, levels have size $< \kappa$ and there is $f : T \to T$ satisfying:

(1) *$f(t) < t$ for $t \in T$ except of the root;*

(2) *for all $t \in T$, $f^{-1}(\{t\})$ is the union of fewer than κ many antichains.*

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Theorem 26 (B., Lopez-Abad, Todorcevic)

If T is a special κ-Aronszajn tree and there are CL-sequences on every $\lambda < \kappa$, then there are CL-sequences on chains of $(T, <_a)$ and $(T, <_c)$. Therefore, there is a CL-sequences on T (hence, on κ).

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Corollary 27

For every infinite cardinal κ below the first Mahlo cardinal, there is a CL-sequence on κ.

Cantor-Bendixson indices

Given a topological space X and $Y \subseteq X$, let Y' be the set of accumulation points of Y. Let $X^{(0)} = X$ and $X^{(\alpha)} = \bigcap_{\beta<\alpha}(X^{(\beta)})'$ for $\alpha > 0$.

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- The Cantor-Bendixson index of X is the smallest ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$.
- If X is compact and scattered, then its Cantor-Bendixson index is the smallest ordinal α such that $X^{(\alpha)} = \emptyset$, so that $\alpha = \beta + 1$ for some β such that $X^{(\beta)}$ is finite.
- For a compact family $\mathcal{F}$, we call β the rank of $\mathcal{F}$ and denote it by $\mathrm{rk}(\mathcal{F})$.
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In the context of families on ω, we have that $\mathrm{rk}([\omega]^{\leq n}) = n$ and $\mathrm{rk}(\mathcal{S}) = \omega$. More complex families are the generalized Schreier families.

Cantor-Bendixson indices

Example 28

A Schreier sequence is defined inductively for $\alpha < \omega_1$ by

1. $\mathcal{S}_0 := [\omega]^{\leq 1}$,
2. $\mathcal{S}_{\alpha+1} := \mathcal{S}_\alpha \otimes \mathcal{S}$
   $= \{\bigcup_{k<n} s_k : n \in \omega,\ s_k < s_{k+1},\ s_k \in \mathcal{S}_\alpha,\ \{\min s_k : k < n\} \in S\}$,
3. $\mathcal{S}_\alpha := \bigcup_{n<\omega}(\mathcal{S}_{\alpha_n} \restriction \omega \setminus n)$ where $(\alpha_n)_n$ is such that $\sup_n \alpha_n = \alpha$, if α is limit;

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Exercise 1

Prove that:

(i) $\bigcup_{n\in\omega} \mathcal{S}_n$ is not compact.

(ii) $\mathcal{S}_\alpha$ is hereditary and compact. Moreover, $\mathrm{rk}(\mathcal{S}_\alpha) = \omega^\alpha$.

Homogeneous families

Fact 29

For every $\alpha < \omega_1$ *and every infinite* $M \subseteq \omega$*, we have that*

$$\mathrm{rk}(\mathcal{S}_\alpha \restriction M) = \mathrm{rk}(\mathcal{S}_\alpha) = \omega^\alpha,$$

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where $\mathcal{F} \restriction M := \mathcal{F} \cap \wp(M)$.
In particular,

$$\mathrm{rk}(\mathcal{S}_\alpha) = \omega^\alpha < \iota(\omega^\alpha) = \iota(\mathrm{rk}(\mathcal{S}_\alpha \restriction M)),$$

where $\iota(\alpha)$ is the smallest exponentially-indecomposable ordinal above α.

Homogeneous families

This motivates the following definitions:

- If $\mathcal{F}$ is a family on some index set I, let

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If $\mathcal{F}$ is a compact homogeneous large family on I, then we get lower and upper bounds for the rank of the collection of the (finite) subsets of indiscernibles of the structure $\mathcal{M}_{\mathcal{F}} := (I, (\mathcal{F} \cap [I]^n)_n)$.

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We improve the previous results and show, for example, the following:

Lemma 30

- *If λ is exp-indecomposable, then*

$$\mathrm{rk}(\mathcal{A}), \mathrm{rk}(\mathcal{C}) < \lambda \Rightarrow \mathrm{rk}(\mathcal{A} \odot \mathcal{C}) < \lambda.$$

- *One of the following holds:*
  - $\mathrm{srk}(\mathcal{A} \odot \mathcal{C}) \leq \mathrm{srk}_a(\mathcal{A})$ *and* $\iota(\mathrm{srk}(\mathcal{A} \odot \mathcal{C})) = \iota(\mathrm{srk}_a(\mathcal{A}))$,
  - $\mathrm{srk}(\mathcal{A} \odot \mathcal{C}) \leq \mathrm{srk}_c(\mathcal{C})$ *and* $\iota(\mathrm{srk}(\mathcal{A} \odot \mathcal{C})) = \iota(\mathrm{srk}_c(\mathcal{C}))$.

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- *Thefore, $\mathcal{A} \odot \mathcal{C}$ is homogeneous, if $\mathcal{A}$ and $\mathcal{C}$ are.*

Bases of homogeneous families

In order to get step up with homogeneous families, we replace the definitions:

- If $\mathcal{F}$ is a family on some index set I, let $\mathrm{srk}(\mathcal{F}) = \min\{\mathrm{rk}(\mathcal{F} \restriction M) : M \text{ is an infinite set of } I\}$.
- A family on I is said to be (α)-homogeneous for some $\omega \leq \alpha < \omega_1$ if $\alpha = \mathrm{srk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{F}) < \iota(\alpha)$.
- A family on I is said to be homogeneous if it is (α)-homogeneous for some $\omega \leq \alpha < \omega_1$.

Bases of homogeneous families

By the following ones:

- If $\mathcal{F}$ is a family on chains of some ordered set P, let $\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) = \min\{\mathrm{rk}(\mathcal{F} \upharpoonright M) : M \text{ is an infinite chain of } P\}$.
- A family on chains of P is said to be $(\alpha, \mathcal{P})$-homogeneous for some $\omega \leq \alpha < \omega_1$ if $\alpha = \mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{F}) < \iota(\alpha)$.
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- A family on chains of P is said to be $\mathcal{P}$-homogeneous if it is $(\alpha, \mathcal{P})$-homogeneous for some $\omega \leq \alpha < \omega_1$.

A family $\mathcal{G}$ on chains of $\mathcal{P}$ is said to be a topological multiplication of a homogeneous family $\mathcal{F}$ on chains of $\mathcal{P}$ by a homogeneous family $\mathcal{H}$ on ω if

- $\mathcal{G}$ is homogeneous and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G})) = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H}))$,
- and $\mathcal{G}$ is a multiplication on chains of $\mathcal{F}$ by $\mathcal{H}$.

Second main result

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(i) $\mathfrak{B}$ is a collection of hereditary, compact and homogeneous families on chains of $\mathcal{P}$ containing all cubes $[P]^{\leq n}_{<}$ and α-homogeneous families for every $\alpha < \omega_1$.

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(ii) $\mathfrak{B}$ is closed under $\cup$ and $\sqcup_{\mathcal{P}}$ and if $\mathcal{F} \subseteq \mathcal{G} \in \mathfrak{B}$ is such that $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})) = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$, then $\mathcal{F} \in \mathfrak{B}$.

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(iii) For every $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H}$ hereditary, compact and homogeneous on ω, there is $\mathcal{G} \in \mathfrak{B}$ which is a topological multiplication of $\mathcal{F}$ by $\mathcal{S}$.

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Theorem 31

If there are bases of homogeneous families on chains of $(T, <_a)$ and $(T, <_c)$, then there is a basis of homogeneous families on T (with any total order).

Second main result

Theorem 32 (B., Lopez-Abad, Todorcevic)

If T is a special κ-Aronszajn tree and there are bases on every $\lambda < \kappa$, then there are bases on chains of $(T, <_a)$ and $(T, <_c)$. Therefore, there is a basis on T (hence, on κ).

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Corollary 33

For every infinite cardinal κ below the first Mahlo cardinal, there is a basis of homogeneous families on κ.

Problems

- What is the minimal cardinal κ such that every (reflexive) Banach space of density κ has a subsymmetric sequence?
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  * Between the first Mahlo and the first ω-Erdös cardinal.
- Characterize (e.g. as colouring principle) κ such that:
  * there is a hereditary and compact (α)-homogeneous family on κ for every $\omega \leq \alpha < \omega_1$.
  * there is a basis of homogeneous families on κ.

Main References

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