Generalizing Schreier families to large index sets II

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Summary of results in Banach spaces

- The Tsirelson space is a *reflexive* Banach space with no subsymmetric sequences.
- (Ketonen, 1974) Any Banach space of density equal to the first ω-Erdös cardinal has subsymmetric sequences.
- (Odell, 1985) There is a Banach space of density $\mathfrak{c}$ with no subsymmetric sequences.
- (Argyros, Motakis, 2014) There is a reflexive Banach space of density $\mathfrak{c}$ with no subsymmetric sequences.
- (B., Lopez-Abad, Todorcevic) For every κ smaller than the first inaccessible cardinal, there is a reflexive Banach space of density κ with no subsymmetric sequences.
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- (B., Lopez-Abad, Todorcevic) For every κ smaller than the first Mahlo cardinal, there is a reflexive Banach space of density κ with no subsymmetric sequences.
- What is the smallest cardinal κ such that every (reflexive) Banach space of density at least κ has a subsymmetric sequence?

First main result

Our main purpose in this talk is to give elements of the proof of the following:

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- There is a CL-sequence of families on κ iff there is a CL-sequence of families on any index set of cardinality κ.
- If there is a CL-sequence of families on κ, then there is a CL-sequence of families on every $\lambda < \kappa$.

Multiplication of families

Given $\mathcal{F}$ on κ and $\mathcal{H}$ on ω, $\mathcal{G}$ on κ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ if every infinite sequence $(s_n)_n$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that, for every $x \in \mathcal{H}$, $\bigcup_{n \in x} t_n \in \mathcal{G}$.

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Example 14

If $\mathcal{F}$ is a hereditary and compact family on κ, then $\mathcal{G} = \mathcal{F} \sqcup \mathcal{F} \sqcup \cdots \sqcup \mathcal{F}$ is a multiplication of $\mathcal{F}$ by $[\omega]^{\leq n}$, where $\mathcal{F} \sqcup \mathcal{F} = \{s \cup t : s, t \in \mathcal{F}\}$.

A CL-sequence on ω

Example 15

Given hereditary and compact families $\mathcal{F}$ and $\mathcal{F}'$ on ω, let

$$\mathcal{F} \oplus \mathcal{F}' = \{s \cup t : s < t,\ s \in \mathcal{F}',\ t \in \mathcal{F}\},$$

$$\mathcal{F} \otimes \mathcal{F}' = \{\bigcup_{k<n} s_k : n \in \omega,\ s_k < s_{k+1},\ s_k \in \mathcal{F},\ \{\min s_k : k < n\} \in \mathcal{F}'\},$$

and notice that $\mathcal{G} = (\mathcal{F} \otimes \mathcal{S}) \oplus \mathcal{F}$ is a multiplication of $\mathcal{F}$ by $\mathcal{S}$.

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Define inductively:

- $\mathcal{F}_0 = \mathcal{S}$;
- $\mathcal{F}_{n+1} = (\mathcal{S}_n \otimes \mathcal{S}) \oplus \mathcal{S}_n$.

$(\mathcal{F}_n)_n$ is a CL-sequence of families on ω.

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For a fixed set P, we will consider multiplications on chains of families on P with respect to some partial order, but also with respect to any total order on P. In this case, we will say "multiplication" instead of "multiplication on chains".

Some warming up: passing from κ to 2^κ

Given a cardinal κ, let $T = 2^{\leq\kappa}$ be the complete binary tree of height $\kappa + 1$ and let $<$ denote the usual partial order on T and $ht : T \to \kappa + 1$ be the height function.

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Our goal now is to pass from multiplications on chains of $(T, <)$ to multiplications on T (with any total order), meaning that the multiplication of a family $\mathcal{F}$ on T must contain many unions of elements of $\mathcal{F}$ and not only unions within some chain.

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- *If $\mathcal{C}$ is a hereditary and compact, then so is $\mathcal{B}_{\mathcal{C}}$.*
- *If $\mathcal{D}$ is a multiplication on chains of $\mathcal{C}$ by $\mathcal{H}$ on $(T,<)$, then $\mathcal{B}_{\mathcal{D}}$ is a multiplication of $\mathcal{B}_{\mathcal{C}}$ by $\mathcal{H}$ on T (with any total order).*

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Corollary 19

For every cardinal κ below the first inaccessible, there is a CL-sequence $(\mathcal{F}_n)_n$ of families on κ.

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Notice that being a chain in $(T, <_a)$ means being a particular type of antichain with respect to $<$ (which we denote by $<_c$ from now on): a subset of immediate successors of a single node of T.

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where $Is''_t x = \{$immediate successors of t below some element of $x\}$.

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Any infinite set X of a tree T contains either an infinite chain, or an infinite comb, or an infinite fan.

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Proof.

If follows from Ramsey Theorem. □

Combinatorial analysis

Theorem 21

If $\mathcal{A}$ and $\mathcal{C}$ are hereditary and compact, then so is $\mathcal{A} \odot \mathcal{C}$.

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Proof.

Hereditariness is clear. For compacity, let $(\tau_k)_k$ be a sequence in $\mathcal{A} \odot \mathcal{C}$. It is enough to assume that τ_k's are subtrees. Assume it converges to some infinite set τ, which has to be a subtree.

- If τ has an infinite chain C, then $(\tau_k \cap C)_k$ which would converge to C, contradicting the compacity of $\mathcal{C}$.
- If τ contains an infinite fan F with root u, then $(Is''_u \tau_k)_k$ would converge to $Is''_u F$, contradicting the compacity of $\mathcal{A}$.

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□

Now, if $\mathcal{A}_1$ and $\mathcal{C}_1$ are a multiplication of $\mathcal{A}_0$ and $\mathcal{C}_0$ by $\mathcal{S}$ respectively, then we want to find a multiplication of $\mathcal{A}_0 \odot \mathcal{C}_0$ by $\mathcal{S}$. The following result is needed...

Theorem 22 (Canonical form of sequences of subtrees)

Suppose that $(\tau_k)_k$ is a sequence of finite subtrees of T. Then there is a subsequence $(\tau_k)_{k\in M}$ which is a Δ-system of root $\bar{\varrho}$ such that

(1) *For every $i \neq j$ and $k \neq l$ in M one has that*

$$\tau_\infty := (\tau_i, \tau_j)_\infty = (\tau_k, \tau_l)_\infty,$$

where $(\tau_i, \tau_j)_\infty$ is the set of maximal elements u of ϱ with the property that there are $v \in \langle \tau_i \cup \tau_j \rangle$, $t_0 \in \tau_0 \setminus \tau_1$ and $t_1 \in \tau_1 \setminus \tau_0$ with $u \leq v \leq t_0, t_1$.

(2) *Let $u \in \tau_\infty$. For each $i < j$ let $\varpi_{i,j}(u)$ be the (unique) maximal $v \in \langle \tau_i \cup \tau_j \rangle$ with the property that there are $t_0 \in \tau_0 \setminus \tau_1$ and $t_1 \in \tau_1 \setminus \tau_0$ with $u \leq v \leq t_0, t_1$. Then $\varpi_i(u) := \varpi_{i,j}(u) = \varpi_{i,k}(u)$ for every $i < j < k$, and $\varpi_i(u) \leq \varpi_j(u)$ for every $i \leq j$. Moreover, one of the following holds.*

(2.1) *$\varpi_i(u) < \varpi_j(u)$ for every $i < j$ and $\varpi_i(u) \notin \bigcup_k \tau_k$ for every $i < j$.*
(2.2) *$\varpi_i(u) = \varpi_j(u) \notin \bigcup_k \tau_k$ for every i.*
(2.3) *$\varpi_i(u) < \varpi_j(u)$ and $\varpi_i(u) \in \tau_i \setminus \varrho$ for every $i < j$.*
(2.4) *$u = \varpi_i(u) = \varpi_j(u) \in \varrho$ for every $i < j$.*

In other words, if $(\tau_k)_k$ is a sequence of finite subtrees of T, there is a subsequence $(\tau_k)_{k \in M}$ which is a Δ-system of root being black points and...

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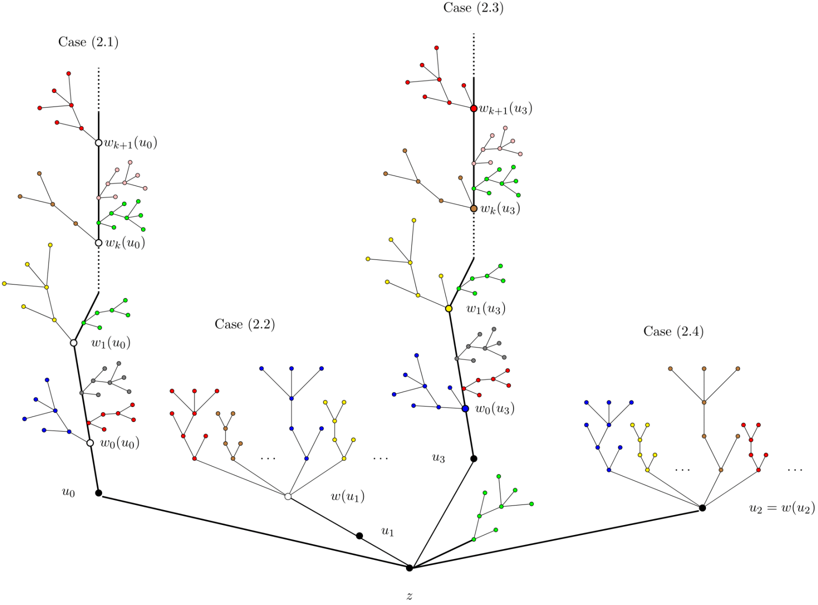

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The proof is a deep analysis of the "new points" and uses Ramsey Theorem many times. The following is a consequence of the previous result.

Theorem 23

If $\mathcal{A}_1$ and $\mathcal{C}_1$ are a multiplication of $\mathcal{A}_0$ and $\mathcal{C}_0$ by $\mathcal{S}$ respectively, then $(\mathcal{A}_1 \sqcup_a [T]^{\leq 1}) \odot (\mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1 \sqcup_c \mathcal{C}_1)$ is a multiplication of $\mathcal{A}_0 \odot \mathcal{C}_0$ by $\mathcal{S}$

Combinatorial analysis

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Corollary 24

If there are CL-sequences on chains of $(T, <_c)$ and of $(T, <_a)$, then there is a CL-sequence on T (with any total order).

First main result

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Theorem 25 (Todorcevic)

For every strongly inaccessible cardinal κ, κ is Mahlo cardinal iff there is no special κ-Aronszajn tree, ie. a tree $(T, <)$ of height κ with no cofinal branches, levels have size $< \kappa$ and there is $f : T \to T$ satisfying:

(1) *$f(t) < t$ for $t \in T$ except of the root;*

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Theorem 26

If T is a special κ-Aronszajn tree and there is a CL-sequence of families on every $\lambda < \kappa$, then there is a CL-sequence of families on T.