Generalizing Schreier families to large index sets

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Winterschool 2017
Outline

1. Introduction
   - Basic notation and definitions
   - Motivation: indiscernibles in Banach spaces

2. First main result
   - Multiplication of families
   - Families on trees
   - Stepping up

3. Second main result
   - Cantor-Bendixson indices and homogeneity
   - Topological multiplication and bases
Main References


Useful tools
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Fact 1

TFAE:

- beer, wine, water, coffee, bread;
- pivo, víno, voda, káva, chléb/chleba.
Basic notation and definitions

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- **pre-compact** if every sequence in $\mathcal{F}$ has a subsequence which forms a $\Delta$-system;
- **large** if it contains arbitrarily large (in cardinality) finite subsets within any infinite subset $X$ of $I$. 
Example 2 (Cubes)

For each $n \in \omega$, the family $[\kappa]^{<n}$ is hereditary and compact, but not large.
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Exercise 1

Let $\mathcal{F}$ be a family on $I$. Prove that:

(i) $\mathcal{F}$ is compact iff every sequence in $\mathcal{F}$ has a subsequence which forms a $\Delta$-system with root in $\mathcal{F}$;
(ii) $\mathcal{F}$ is pre-compact iff $\mathcal{F} \subseteq \{ s \subseteq I : \exists t \in \mathcal{F}, s \subseteq t \}$ is compact;
(iii) if $\mathcal{F}$ is hereditary, then it is compact iff it is pre-compact;
(iv) if $\mathcal{F}$ is compact, then $\mathcal{F}$ is scattered.

Example 3 (Schreier family)

The family $S = \{ \emptyset \} \cup \{ s \in [\omega]^{<\omega} : |s| \leq \min s + 1 \}$ is hereditary, compact and large.
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Indiscernibles

In model theory, a set of indiscernibles for a given structure $\mathcal{M}$ is a subset $X$ with a total order $<$ such that, for every positive integer $n$, every two increasing $n$-tuples $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ of elements of $X$ have the same properties in $\mathcal{M}$. 

Proposition 1

If $F$ is a compact large family on $I$, then the relational structure $\mathcal{M}_F := (I, (F \cap [I]^n))_n$ has no infinite sets of indiscernibles.

Proof.

Suppose $X \subseteq I$ is infinite and $(X, <)$ is a set of indiscernibles. Given $n \geq 1$, since $F$ is large, there is $s \in [X]^n \cap F$.

Now, given $t \in [X]^n$, writing $s = \{x_1 < \cdots < x_n\}$ and $t = \{y_1 < \cdots < y_n\}$, we get that $t \in [X]^n \cap F$.

Hence, $[X] < \omega \subseteq F$, contradicting the fact that $F$ is compact.
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Indiscernibles in Banach spaces: subsymmetric sequences

A sequence \((x_n)_n\) in a Banach space \(X\) is subsymmetric if there is \(C \geq 1\) such that for all \((\lambda_i)_{i=1}^l\) and all increasing sequences \((k_i)_{i=1}^l\) and \((n_i)_{i=1}^l\) we have that

\[
\| \sum_{i=1}^l \lambda_i x_{k_i} \| \leq C \| \sum_{i=1}^l \lambda_i x_{n_i} \|.
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Example 4

The unit bases of \(c_0\) and \(\ell_p\), \(1 \leq p < \infty\) are (sub)symmetric.
Example 5 (Schreier space)

Given: \( x = (x_n)_n \in c_{00}(\omega) \), let \( \| x \|_S = \sup \{ \sum_{n \in s} |x_n| : s \in S \} \).

\( \| \cdot \|_S \) is a norm and the completion of \( (c_{00}(\omega), \| \cdot \|_S) \) is a Banach space such that \( (e_n)_n \) is an unconditional basis with no subsymmetric basic subsequences.
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- $[\omega]^{\leq 1} \subseteq S \Rightarrow \|\cdot\|_\infty \leq \|\cdot\|_S \Rightarrow \|\cdot\|_S$ is a norm;
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- $[\omega]^{< 1} \subseteq S \Rightarrow \|\cdot\|_{\infty} \leq \|\cdot\|_S \Rightarrow \|\cdot\|_S$ is a norm;
- hereditariness of $\mathcal{F} \Rightarrow$ projections on the first $m$-many coordinates are bounded $\Rightarrow (e_n)_n$ is a Schauder basis, clearly unconditional;
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- \([\omega]^{\leq 1} \subseteq S \Rightarrow \| \cdot \|_\infty \leq \| \cdot \|_S \Rightarrow \| \cdot \|_S \) is a norm;
- hereditariness of \( F \Rightarrow \) projections on the first \( m \)-many coordinates are bounded \( \Rightarrow (e_n)_n \) is a Schauder basis, clearly unconditional;
- compactness + Ptak’s Lemma + largeness of \( F \Rightarrow (e_n)_n \) has no subsymmetric subsequences.

Lemma 6 (Pták, 1963)

If \( F \) is a compact family on \( \omega \), then for every \( \varepsilon > 0 \), there is a finite \( F \subseteq \omega \) and positive \((a_\alpha)_{\alpha \in F}\) such that \( \sum_{\alpha \in F} a_\alpha = 1 \) and \( \sum_{\alpha \in s} a_\alpha < \varepsilon \) if \( s \in F \cap \varnothing(F) \).
Lopez-Abad and Todorcevic result

Theorem 7 (Lopez-Abad, Todorcevic, 2013)

Let $\kappa$ be an infinite cardinal. TFAE:

(a) $\kappa$ is not $\omega$-Erdös, i.e., if $\kappa \not\rightarrow (\omega)^<\omega$;

(b) there is a hereditary, compact and large family $\mathcal{F}$ on $\kappa$;

(c) there is a nontrivial normalized weakly-null basis $(x_\alpha)_{\alpha < \kappa}$ in a Banach space with no subsymmetric basic subsequence.
(a) implies (b)

Fact 8

If \( \kappa \not\rightarrow (\omega)_2^{<\omega} \) and \( c : [\kappa]^{<\omega} \rightarrow 2 \), then

\[
\mathcal{F}_c = \{ s \subseteq \omega : s \text{ is monochromatic} \}
\]

is a hereditary, compact and large family on \( \kappa \).
Fact 8

If $\kappa \nrightarrow (\omega)^{<\omega}_2$ and $c : [\kappa]^{<\omega} \rightarrow 2$, then

$$\mathcal{F}_c = \{ s \subseteq \omega : s \text{ is monochromatic} \}$$

is a hereditary, compact and large family on $\kappa$.

Proof.

It is clearly hereditary and it is easy to check that it is compact. Largeness is a consequence of the finite Ramsey theorem. The fact that $\kappa \nrightarrow (\omega)^{<\omega}_2$ is needed only to guarantee that $\mathcal{F}_c$ consists of finite subsets of $\kappa$. \qed
Fact 9

If $\mathcal{F}$ is a hereditary, compact and large family on $\kappa$ and $x = (x_\alpha)_\alpha \in c_{00}(\kappa)$, let

$$\|x\|_{\mathcal{F}} = \sup\left\{ \sum_{\alpha \in s} |x_\alpha| : s \in \mathcal{F} \right\}.$$ 

$\| \cdot \|_{\mathcal{F}}$ is a norm and the completion of $(c_{00}(\kappa), \| \cdot \|_{\mathcal{F}})$ is a Banach space such that $(e_\alpha)_\alpha$ is an unconditional basis with no subsymmetric basic subsequences.
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Proof.

Analogous to the Schreier space.
Exercise 2

\[ \kappa \rightarrow (\omega)^{<\omega}_2 \iff \kappa \rightarrow (\omega)^{<\omega}_2. \]

**Hint:** Given \( c : [\kappa]^{<\omega} \rightarrow 2^\omega \) and \( \theta : \omega^2 \rightarrow \omega \) bijection such that \( \theta(i, j) \geq i \), let \( d : [\kappa]^{<\omega} \rightarrow 2 \) be such that \( d(s) \) is the \( j \)-th coordinate of the \( c \)-color of the subset of \( s \) consisting of its first \( i \)-many elements, where \( \theta(i, j) = |s| \) and show that a \( d \)-monochromatic set is also \( c \)-monochromatic.
(c) implies (a)

**Exercise 2**

\[ \kappa \rightarrow (\omega)^{<\omega}_2 \text{ iff } \kappa \rightarrow (\omega)^{<\omega}_2. \]

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**Fact 10 (Ketonen, 1974)**

Given \( (x_\alpha)_{\alpha < \kappa} \), for each \( s = \{ \alpha_1 < \cdots < \alpha_n \} \in [\kappa]^{<\omega} \) with \( |s| = n \), define \( f_s \) on \( \mathbb{R}^n \) by \( f_s(a_1, \ldots, a_n) = \|a_1 x_{\alpha_1} + \cdots + a_n x_{\alpha_n}\| \) and define \( c : [\kappa]^{<\omega} \rightarrow \bigcup_{n \in \omega} \{n\} \times \mathbb{R}^{n+1} \) by \( c(s) = (|s|, f_s) \). If \( A \) is an infinite monochromatic subset of \( \kappa \), then \( (x_\alpha)_{\alpha \in A} \) is symmetric.
Tsirelson space

Let us now turn to the “full” (in contrast with the “sequential”) version of the problem, i.e., whether there is a Banach space with no subsymmetric basic sequences.
Tsirelson space

Let us now turn to the “full” (in contrast with the “sequential”) version of the problem, i.e., whether there is a Banach space with no subsymmetric basic sequences. The first such example was given by Tsirelson.

Example 11 (Tsirelson space)

Given $x = (x_n)_n \in c_{00}(\omega)$, let $\|x\|_T$ on $c_{00}(\omega)$ be such that

$$\|x\|_T = \sup\{\|x\|_\infty, \frac{1}{2} \sum_{i=1}^{n} \|\langle x_i, \chi_{s_i} \rangle\|_T : s_i < s_{i+1}, \{\min s_i\}_{1 \leq i \leq n} \in S\}.$$ 

$\| \cdot \|_T$ is a norm and the completion of $(c_{00}(\omega), \| \cdot \|_T)$ is a (separable) reflexive Banach space with no subsymmetric basic sequences.
Nonseparable Tsirelson-like spaces

However, the natural nonseparable version of the Tsirelson space, replacing the Schreier family by a hereditary compact and large family on an uncountable cardinal $\kappa$, yields a space with copies of $\ell_1$, hence with subsymmetric basic sequences (Lopez-Abad, Todorcevic, 2013).
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Fact 12

*If $\mathcal{F}$ is a large and spreading family on an uncountable index set, then $\mathcal{F}$ is not compact.*
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Fact 12

*If $F$ is a large and spreading family on an uncountable index set, then $F$ is not compact.*

To overcome this obstacle, we switch from a single large family to sequences of families obtained by making some kind of products by families on $\omega$, such as the Schreier family.
Nonseparable Tsirelson-like spaces

Given a family $\mathcal{F}$ on a cardinal $\kappa$ and a family $\mathcal{H}$ on $\omega$, we say that a family $\mathcal{G}$ on $\kappa$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ if every infinite sequence $(s_n)_n$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that, for every $x \in \mathcal{H}$, $\bigcup_{n \in x} t_n \in \mathcal{G}$. 

Theorem 13 (B., Lopez-Abad, Todorcevic)

For every infinite cardinal $\kappa$ smaller than the first Mahlo cardinal, there is a CL-sequence of families on $\kappa$. 

Recall that a cardinal $\kappa$ is Mahlo if it is strongly inaccessible and $\{\lambda < \kappa : \lambda \text{ is strongly inaccessible}\}$ is stationary.
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We say that a sequence of families $(\mathcal{F}_n)_n$ on $\kappa$ is a CL-sequence (consecutively large sequence) of families on $\kappa$ if each family is hereditary and compact and $\mathcal{F}_{n+1}$ is a multiplication of $\mathcal{F}_n$ by $\mathcal{S}$.
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Recall that a cardinal $\kappa$ is Mahlo if it is strongly inaccessible and $\{\lambda < \kappa : \lambda \text{ is strongly inaccessible}\}$ is stationary.
Nonseparable Tsirelson-like spaces

Given a family $\mathcal{F}$ on a cardinal $\kappa$ and a family $\mathcal{H}$ on $\omega$, we say that a family $\mathcal{G}$ on $\kappa$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ if every infinite sequence $(s_n)_n$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that, for every $x \in \mathcal{H}$, $\bigcup_{n \in x} t_n \in \mathcal{G}$.

We say that a sequence of families $(\mathcal{F}_n)_n$ on $\kappa$ is a CL-sequence (consecutively large sequence) of families on $\kappa$ if each family is hereditary and compact and $\mathcal{F}_{n+1}$ is a multiplication of $\mathcal{F}_n$ by $S$.

**Theorem 13 (B., Lopez-Abad, Todorcevic)**

*For every infinite cardinal $\kappa$ smaller than the first Mahlo cardinal, there is a CL-sequence of families on $\kappa$.*

Recall that a cardinal $\kappa$ is **Mahlo** if it is strongly inaccessible and $\{\lambda < \kappa : \lambda \text{ is strongly inaccessible}\}$ is stationary.
Nonseparable Tsirelson-like spaces

Theorem 14 (B., Lopez-Abad, Todorcevic & Argyros, Motakis)

If \((F_n)_n\) is a CL-sequence, then there is a Banach space \(X\) of density \(\kappa\) with an unconditional (long) basis and with no subsymmetric sequences.
Theorem 14 (B., Lopez-Abad, Todorcevic & Argyros, Motakis)

If \((F_n)_n\) is a CL-sequence, then there is a Banach space \(X\) of density \(\kappa\) with an unconditional (long) basis and with no subsymmetric sequences.

Sketch.

Given \(x \in c_{00}(\kappa)\), let

\[
\|x\| = \sup\{\|x\|_\infty, \sum_{n=0}^{\infty} \frac{\|x\|_{F_n}}{2^{n+1}} \|T\}\}.
\]

This is a norm such that the closure with respect to it is a Banach space of density \(\kappa\) with an unconditional basis and with no subsymmetric sequences.
A CL-sequence on $\omega$

Example 15

Given hereditary and compact families $\mathcal{F}$ and $\mathcal{F}'$ on $\omega$, let

$$\mathcal{F} \oplus \mathcal{F}' = \{ s \cup t : s < t, \ s \in \mathcal{F}', \ t \in \mathcal{F} \},$$

$$\mathcal{F} \otimes \mathcal{F}' = \bigcup_{k < n} s_k : n \in \omega, \ s_k < s_{k+1}, \ s_k \in \mathcal{F}, \ \{ \min s_k : k < n \} \in \mathcal{F}' \},$$

and notice that $\mathcal{G} = (\mathcal{F} \otimes S) \oplus \mathcal{F}$ is a compact and hereditary family on $\omega$ and a multiplication of $\mathcal{F}$ by $S$. 
A CL-sequence on $\omega$

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and notice that $\mathcal{G} = (\mathcal{F} \otimes S) \oplus \mathcal{F}$ is a compact and hereditary family on $\omega$ and a multiplication of $\mathcal{F}$ by $S$.

Define inductively:

- $\mathcal{F}_0 = S$;
- $\mathcal{F}_{n+1} = (S_n \otimes S) \oplus S_n$.

$(\mathcal{F}_n)_n$ is a CL-sequence of families on $\omega$. 