On measurable *Hamel functions*

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A Hamel base

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1. A basis of $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$ is called Hamel basis.
2. 1905, Georg Hamel, used this notion to obtain the existence of a discontinuous solutions of the Cauchy equation:
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Marczewski field and sets

1. $A \in (s)$ iff $\forall_{P \in Perf} \exists_{Q \in Perf}\ Q \subseteq A \vee Q \cap A = \emptyset$.
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Notation

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- Suppose that $f : \mathbb{R} \to \mathbb{R}$.
- We say that f is a Hamel function if f, considered as a subset of $\mathbb{R}^2$, is a Hamel basis of $\mathbb{R}^2$.

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The notion of a Hamel function

K.Płotka
On functions whose graph is a Hamel basis.
Proc. Amer. Math. Soc.Vol. 131, No 4, (2003), 1031 – 1041.

Hamel functions examined...

What is known about Hamel functions?

- There exists such a function!
- Theorem: (K.Płotka) Every $f : \mathbb{R} \to \mathbb{R}$ is the pointwise sum of two Hamel functions.

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The base article

Rafał Filipów, Andrzej Nowik, Piotr Szuca
There are measurable Hamel functions.
Submitted

Main results

Theorem:

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- $\exists_{B\in\mathcal{I}}$ and a Hamel basis $H \subset B$ with $|B \setminus H| = 2^\omega$.
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Definition of σ–porous sets

Theorem:

- $$p(X, r) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(X, (r - \varepsilon, r + \varepsilon))}{\varepsilon},$$

  where $\lambda(X, I)$ denotes the maximal length of an open subinterval of the interval I which is disjoint from X.
- X is *porous* ($X \in \mathcal{P}$) iff $\forall_{a \in X} p(X, a) > 0$.
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Thank You
for
Your Attention