On measurable *Hamel functions*

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1 Basic notions and the (Pre)history

2 Introduction

3 The origin of the notion of a Hamel function

4 The next results base on:

5 The proof machine

6 Further results
A Hamel base

1. A basis of $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$ is called Hamel basis.

2. 1905, Georg Hamel, used this notion to obtain the existence of a discontinuous solutions of the Cauchy equation:

3. $f(x + y) = f(x) + f(y)$
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σ – fields

Marczewski field and sets

1. \( A \in (s) \) iff \( \forall P \in \text{Perf} \exists Q \in \text{Perf} \ Q \subseteq A \lor Q \cap A = \emptyset \).

2. \( A \in (s_0) \) iff \( \forall P \in \text{Perf} \exists Q \in \text{Perf} \ Q \cap A = \emptyset \).

Marczewski measurable function

- \( f : \mathbb{R} \to \mathbb{R} \) is Marczewski measurable iff \( \forall U \text{ open} \Rightarrow f^{-1}[U] \in (s) \).
- \( f : \mathbb{R} \to \mathbb{R} \) is Marczewski measurable iff \( \forall P \in \text{Perf} \exists Q \in \text{Perf} \ f \mid Q \) is continuous.
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- Suppose that \( f : \mathbb{R} \to \mathbb{R} \).
- We say that \( f \) is a Hamel function if \( f \), considered as a subset of \( \mathbb{R}^2 \), is a Hamel basis of \( \mathbb{R}^2 \).

Who and whene introduced this notion?

- The class of Hamel functions was introduced by K. Płotka in...
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K. Płotka

On functions whose graph is a Hamel basis.

What is known about Hamel functions?

- There exists such a function!
- Theorem: (K. Płotka) Every $f : \mathbb{R} \to \mathbb{R}$ is the pointwise sum of two Hamel functions.
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*There are measurable Hamel functions.*

Submitted
Theorem:

- There exists a **Marczewski measurable** Hamel function.
- There exists a **Lebesgue measurable** Hamel function.
- There exists a Hamel function with **the Baire property**.
Main results

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The proof machine

Theorem:

- $I$ is a $\sigma$-ideal of subsets of $\mathbb{R}$ which contains singletons.
- $\exists B \in I$ and a Hamel basis $H \subset B$ with $|B \setminus H| = 2^\omega$.
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Theorem:

- $\mathcal{I}$ is a Borel generated (ccc) $\sigma$-ideal of subsets of $\mathbb{R}$ which contains singletons.
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On measurable Hamel functions

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Definition of $\sigma$–porous sets

**Theorem:**

$$p(X, r) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(X, (r - \varepsilon, r + \varepsilon))}{\varepsilon},$$

where $\lambda(X, I)$ denotes the maximal length of an open subinterval of the interval $I$ which is disjoint from $X$.

- $X$ is porous ($X \in \mathcal{P}$) iff $\forall a \in X \; p(X, a) > 0$.
- $\sigma \mathcal{P}$ denote the sigma-ideal generated by the porous sets.
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Further results

Theorem:

- There exists a Hamel function which is measurable with respect to the $\sigma$-field $\Bor \bigtriangleup \sigma \mathcal{P}$ (sigma porous sets).
- There exists a Hamel function which is measurable with respect to the $\sigma$-field $\Bor \bigtriangleup \mathcal{E}$.
- There exists a Hamel function which is measurable with respect to the $\sigma$-field $\Bor \bigtriangleup (\mathcal{N} \cap \mathcal{M})$. 
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Thank You for Your Attention

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