On weakly Radon-Nikodým compact spaces

Winter School in Abstract Analysis

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Table of Contents

1. Weakly Radon-Nikodým compact spaces

2. Stability under continuous images

3. Existence of convergent sequences
Definition (E. Glasner and M. Megrelishvili)

A compact space $K$ is said to be weakly Radon-Nikodým (WRN for short) if it is homeomorphic to a weak*-compact subset of the dual of a Banach space not containing an isomorphic copy of $\ell_1$. 
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Definition

- A set $S \subset X$ is said to be weakly precompact if every sequence in $S$ has a weakly Cauchy subsequence.
**Definition (E. Glasner and M. Megrelishvili)**

A compact space $K$ is said to be **weakly Radon-Nikodým** (WRN for short) if it is homeomorphic to a weak*-compact subset of the dual of a Banach space not containing an isomorphic copy of $\ell_1$.

**Definition**

- A set $S \subset X$ is said to be **weakly precompact** if every sequence in $S$ has a weakly Cauchy subsequence.

- $X$ is **weakly precompactly generated** (WPG) if there exists a weakly precompact set $S \subset X$ such that $\overline{\text{span}} S = X$. 
... By analogy with the well-known class of weakly compactly generated Banach spaces, one may call a Banach space such as $X$ above a weakly precompactly generated (or WPG) space. The above example shows that WPG spaces exhibit certain pathologies that do not occur for WCG spaces, and indeed do not occur for the interesting ‘WKA spaces’ of Talagrand. The present author would be interested to know whether WPG spaces have any of the good properties of these other classes, and whether there is a nice characterization of those compact spaces $T$ for which $C(T)$ is WPG. One obvious question is whether every such space $T$ contains a nontrivial convergent sequence.

R. Haydon, 1980
Lemma

If $K$ is WRN then $C(K)$ is WPG.
Lemma

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Proof.

- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$. 
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If $K$ is WRN then $C(K)$ is WPG.

Proof.

- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$.
- Then, $B_X$ is weakly precompact in $X$. 
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If $K$ is WRN then $C(K)$ is WPG.

Proof.

- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$.
- Then, $B_X$ is weakly precompact in $X$.
- Each $x \in B_X$ defines a continuous function $x : K \to \mathbb{R}$ such that $x(x^*) = x^*(x)$ for every $x^* \in K$. 
Lemma

If $K$ is WRN then $\mathcal{C}(K)$ is WPG.

Proof.

- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$.
- Then, $B_X$ is weakly precompact in $X$.
- Each $x \in B_X$ defines a continuous function $x : K \to \mathbb{R}$ such that $x(x^*) = x^*(x)$ for every $x^* \in K$.
- $B_X \subset \mathcal{C}(K)$ is weakly precompact in $\mathcal{C}(K)$. 
Lemma

If $K$ is WRN then $C(K)$ is WPG.

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- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$.
- Then, $B_X$ is weakly precompact in $X$.
- Each $x \in B_X$ defines a continuous function $x : K \to \mathbb{R}$ such that $x(x^*) = x^*(x)$ for every $x^* \in K$.
- $B_X \subset C(K)$ is weakly precompact in $C(K)$.
- $W = \text{co}(B_X \cup \{-1, 1\})$ is weakly precompact in $C(K)$.
Lemma

If $K$ is WRN then $C(K)$ is WPG.

Proof.

- WLOG, $K \subset B_{X^*}$ with $X$ not containing $\ell_1$.
- Then, $B_X$ is weakly precompact in $X$.
- Each $x \in B_X$ defines a continuous function $x : K \to \mathbb{R}$ such that $x(x^*) = x^*(x)$ for every $x^* \in K$.
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- $W = \text{co}(B_X \cup \{-1, 1\})$ is weakly precompact in $C(K)$.
- $L = \sum \frac{W^n}{2^n}$ is weakly precompact and $\text{span} L = C(K)$. 

\[\square\]
Lemma

If $X$ is WPG then $B_{X^*}$ is WRN.

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Proof.

- If $X$ is WPG, then there exists a Banach space $Y$ not containing $\ell_1$ and a bounded linear operator $T : Y \to X$ with dense range (Davis-Figiel-Johnson-Pelczyński Factorization Theorem).
Lemma

If $X$ is WPG then $B_{X^*}$ is WRN.

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- If $X$ is WPG, then there exists a Banach space $Y$ not containing $\ell_1$ and a bounded linear operator $T : Y \to X$ with dense range (Davis-FIGiel-Johnson-Pelczynski Factorization Theorem).

- $T^* : X^* \to Y^*$ restricted to $B_{X^*}$ is an embedding from $B_{X^*}$ into the dual of a Banach space not containing $\ell_1$. 
**Lemma**

- A compact space $K$ is WRN if and only if $C(K)$ is WPG.
- If $X$ is a WPG Banach space, then $B_{X^*}$ is WRN.
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- If $X$ is a WPG Banach space, then $B_{X^*}$ is WRN.

Theorem (H. Rosenthal, 1974)

There exists a non-WPG Banach space $X$ such that $B_{X^*}$ is WRN.
Definition

- **K is Eberlein** if it is homeomorphic to a weak-compact space of a Banach space.
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- **K is Corson** if it is homeomorphic to a compact subspace of
  \[ \Sigma(\Gamma) = \{ x \in \mathbb{R}^\Gamma : \text{supp } x \text{ is countable} \} \] for some set \( \Gamma \).
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- **K is in the class MS** if every Radon measure on \( K \) is separable.
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- **X is weakly compactly generated (WCG)** if there exists a weakly compact set \( W \subset X \) such that \( \overline{\text{span}} W = X \).
- **X is weakly Lindelöf determined (WLD)** if \( (X^*, \omega^*) \) can be embedded in \( \Sigma(\Gamma) \) for some set \( \Gamma \).
Definition

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- **K is Radon-Nikodým (RN)** if it is homeomorphic to a weak$^*$-compact subset of the dual $X^*$ of an Asplund space $X$.
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- **X is Asplund generated** if there exist an Asplund space $Y$ and a bounded linear operator $T : Y \to X$ with dense range.
Lemma

- *K is Eberlein if and only if* \( \mathcal{C}(K) \) *is WCG;*
Lemma

- \( K \) is Eberlein if and only if \( C(K) \) is WCG;
- If \( X \) is WCG then \( B_{X^*} \) is Eberlein;
Lemma

- $K$ is Eberlein if and only if $C(K)$ is WCG;
- If $X$ is WCG then $B_{X^*}$ is Eberlein;
- $K$ is RN if and only if $C(K)$ is Asplund generated;
Lemma

- $K$ is Eberlein if and only if $C(K)$ is WCG;
- If $X$ is WCG then $B_{X^*}$ is Eberlein;
- $K$ is RN if and only if $C(K)$ is Asplund generated;
- If $X$ is Asplund generated then $B_{X^*}$ is RN;
Lemma

- $K$ is Eberlein if and only if $C(K)$ is WCG;
- If $X$ is WCG then $B_{X^*}$ is Eberlein;
- $K$ is RN if and only if $C(K)$ is Asplund generated;
- If $X$ is Asplund generated then $B_{X^*}$ is RN;
- $K$ is Corson and is in the class MS if and only if $C(K)$ is WLD;
Lemma

- $K$ is Eberlein if and only if $C(K)$ is WCG;
- If $X$ is WCG then $B_{X^*}$ is Eberlein;
- $K$ is RN if and only if $C(K)$ is Asplund generated;
- If $X$ is Asplund generated then $B_{X^*}$ is RN;
- $K$ is Corson and is in the class MS if and only if $C(K)$ is WLD;
- $X$ is WLD if and only if $B_{X^*}$ is Corson;
$K$ is Eberlein if and only if it can be embedded in $c_0(\Gamma)$ for some set $\Gamma$ (D. Amir and J. Lindenstrauss, 1968);
\begin{itemize}
\item $K$ is Eberlein if and only if it can be embedded in $c_0(\Gamma)$ for some set $\Gamma$ (D. Amir and J. Lindenstrauss, 1968);
\item $K$ is RN if and only if there is a l.s.c. metric on $K$ which fragments $K$ (I. Namioka, 1987);
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  \begin{itemize}
  \item A metric $d$ fragments $K$ if for every $\varepsilon > 0$ and every closed $F \subset K$ there is an open $U \subset K$ such that $U \cap F \neq \emptyset$ and $\text{diam}_d(U \cap F) < \varepsilon$;
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  - $d$ is l.s.c. if for every distinct points $x, y \in K$ and $d(x, y) > \delta > 0$ there are open sets $x \in U$ and $y \in V$ such that $d(U, V) > \delta$;
Definition

A family $\mathcal{F} \subset C(K)$ is **fragmented** if for every $\varepsilon > 0$ and every closed set $F \subset K$ there is an open $U \subset K$ such that $U \cap F \neq \emptyset$ and $\text{diam}_f(U \cap F) < \varepsilon$ for every $f \in \mathcal{F}$;
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**Theorem (H. Rosenthal, E. Glasner and M. Megrelishvili)**

Let $K$ be a compact space and $\mathcal{F} \subset \mathcal{C}(K)$ uniformly bounded.

TFAE:
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Theorem (H. Rosenthal, E. Glasner and M. Megrelishvili)

Let $K$ be a compact space and $\mathcal{F} \subset C(K)$ uniformly bounded. TFAE:

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Let $K$ be a compact space and $\mathcal{F} \subset C(K)$ uniformly bounded. TFAE:

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Theorem (E. Glasner and M. Megrelishvili, 2012)

*K is WRN if and only if there exists an eventually fragmented uniformly bounded family $\mathcal{F} \subset C(K)$ which separates the points of $K$.*
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\( K \) is WRN if and only if there exists an eventually fragmented uniformly bounded family \( \mathcal{F} \subset \mathcal{C}(K) \) which separates the points of \( K \).

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Proof.

\( \Rightarrow \): 

- \( K \) WRN \( \Rightarrow \) \( \mathcal{C}(K) \) WPG \( \Rightarrow \) there exists \( \mathcal{F} \subset \mathcal{C}(K) \) weakly precompact such that \( \overline{\text{span}} \mathcal{F} = \mathcal{C}(K) \).
Theorem (E. Glasner and M. Megrelishvili, 2012)

$K$ is WRN if and only if there exists an eventually fragmented uniformly bounded family $\mathcal{F} \subset C(K)$ which separates the points of $K$.

Proof.

$\Rightarrow$:

- $K$ WRN $\Rightarrow$ $C(K)$ WPG $\Rightarrow$ there exists $\mathcal{F} \subset C(K)$ weakly precompact such that $\overline{\text{span}} \mathcal{F} = C(K)$.

- $\mathcal{F}$ is an eventually fragmented uniformly bounded family which separates the points of $K$. 
Theorem (E. Glasner and M. Megrelishvili, 2012)

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Proof.

\( \Rightarrow \):

- \( K \text{ WRN} \Rightarrow \mathcal{C}(K) \text{ WPG} \Rightarrow \text{there exists } \mathcal{F} \subset \mathcal{C}(K) \text{ weakly precompact such that } \overline{\text{span}} \mathcal{F} = \mathcal{C}(K) \).

- \( \mathcal{F} \) is an eventually fragmented uniformly bounded family which separates the points of \( K \).
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- \( \mathcal{F} \subset \mathcal{C}(K) \) an eventually fragmented uniformly bounded family which separates the points of \( K \).
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- $L = \sum \frac{W^n}{2^n}$ is weakly precompact and $\overline{\text{span}} \ L = C(K)$. 

\boxed{\text{Q.E.D.}}
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Question (E. Glasner and M. Megrelishvili)

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A compact space $K$ is said to be quasi RN (QRN) if and only if there is a Reznichenko metric on $K$ which fragments $K$. 
Question (I. Namioka, 1987)

Is the continuous image of a RN compact space also RN?

Definition

A compact space $K$ is said to be **quasi RN (QRN)** if and only if there is a Reznichenko metric on $K$ which fragments $K$. 

- $d$ is said to be Reznichenko if for every distinct points $x, y \in K$ there are open sets $x \in U$ and $y \in V$ such that $d(U, V) > 0$. 

Theorem (A. D. Arvanitakis, 2002)

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Definition

A sequence \((A_n^0, A_n^1)_{n \in \mathbb{N}}\) of disjoint pairs of subsets of a set \(S\) is said to be independent if for every natural number \(n\) and every \(\varepsilon: \{1, 2, \ldots, n\} \rightarrow \{0, 1\}\) we have \(\bigcap_{k=1}^{n} A_{\varepsilon(k)}^k \neq \emptyset\).
Definition

- A sequence \((A^0_n, A^1_n)_{n \in \mathbb{N}}\) of disjoint pairs of subsets of a set \(S\) is said to be **independent** if for every natural number \(n\) and every \(\varepsilon : \{1, 2, ..., n\} \rightarrow \{0, 1\}\) we have \(\bigcap_{k=1}^{n} A^\varepsilon(k) \neq \emptyset\).

- A sequence of functions \((f_n)_{n=1}^{\infty} \subset \mathbb{R}^S\) is said to be **independent** if there are \(p < q\) such that the sequence \((A^0_n, A^1_n)_{n \in \mathbb{N}}\) is independent, where \(A^0_n = \{s : f_n(s) < p\}\) and \(A^1_n = \{s : f_n(s) > q\}\) for every natural number \(n\).
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- A sequence \((A_0^n, A_1^n)_{n \in \mathbb{N}}\) of disjoint pairs of subsets of a set \(S\) is said to be independent if for every natural number \(n\) and every \(\varepsilon : \{1, 2, \ldots, n\} \to \{0, 1\}\) we have \(\bigcap_{k=1}^n A_{\varepsilon(k)}^\varepsilon \neq \emptyset\).

- A sequence of functions \((f_n)_{n=1}^\infty \subset \mathbb{R}^S\) is said to be independent if there are \(p < q\) such that the sequence \((A_0^n, A_1^n)_{n \in \mathbb{N}}\) is independent, where \(A_0^n = \{s : f_n(s) < p\}\) and \(A_1^n = \{s : f_n(s) > q\}\) for every natural number \(n\).

Remark

A compact space \(K\) is WRN if and only if there exists a set \(\Gamma\) such that \(K \hookrightarrow [0, 1]^\Gamma\) and for every \(p < q\), the family of pairs \(A_\alpha^0 = \{x \in K : x_\alpha < p\}\), \(A_\alpha^1 = \{x \in K : x_\alpha > q\}\) with \(\alpha \in \Gamma\) does not contain independent sequences.
Definition

A compact space $K \hookrightarrow [0, 1]^\Gamma$ is quasi-WRN (QWRN for short) if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n^\varepsilon$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $A_{\alpha}^0 = \{x \in K : x_\alpha < p\}$, $A_{\alpha}^1 = \{x \in K : x_\alpha > q\}$ with $\alpha \in \Gamma_n^\varepsilon$ does not contain independent sequences.


**Definition**

A compact space $K \hookrightarrow [0,1]^\Gamma$ is **quasi-WRN** (QWRN for short) if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma^\varepsilon_n$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $A^0_\alpha = \{x \in K : x_\alpha < p\}$, $A^1_\alpha = \{x \in K : x_\alpha > q\}$ with $\alpha \in \Gamma^\varepsilon_n$ does not contain independent sequences.

**Lemma**

Every WRN compact space is QWRN.
**Definition**

A compact space $K \hookrightarrow [0,1]^\Gamma$ is **quasi-WRN** (QWRN for short) if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n^\varepsilon$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $A^0_\alpha = \{x \in K : x_\alpha < p\}$, $A^1_\alpha = \{x \in K : x_\alpha > q\}$ with $\alpha \in \Gamma_n^\varepsilon$ does not contain independent sequences.

**Lemma**

Every WRN compact space is QWRN.

**Theorem**

The definition of being QWRN does not depend on the set $\Gamma$. 
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- WLOG, $K \subset \{0, 1\}^\Gamma$ for some set $\Gamma$. 
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- WLOG, \( K \subset \{0, 1\}^\Gamma \) for some set \( \Gamma \).
- \( K \) QWRN \( \Rightarrow \) There exists a decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n \) such that for every \( p < q \) with \( q - p > \frac{1}{2} \), the family of pairs \( A^0_\alpha = \{ x \in K : x_\alpha < p \} \), \( A^1_\alpha = \{ x \in K : x_\alpha > q \} \) with \( \alpha \in \Gamma_n \) does not contain independent sequences.

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- Let $\mathcal{F} = \{ f_\alpha \}_{\alpha \in \Gamma} \subset C(K)$, where $f_\alpha(x) = \frac{x_\alpha}{n}$ for every $n \in \mathbb{N}$, $\alpha \in \Gamma_n$ and $x \in K$. 
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- **WLOG**, \( K \subseteq \{0, 1\}^\Gamma \) for some set \( \Gamma \).
- \( K \) QWRN \( \Rightarrow \) There exists a decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n \) such that for every \( p < q \) with \( q - p > \frac{1}{2} \), the family of pairs \( A^0_\alpha = \{x \in K : x_\alpha < p\}, A^1_\alpha = \{x \in K : x_\alpha > q\} \) with \( \alpha \in \Gamma_n \) does not contain independent sequences.
- Let \( \mathcal{F} = \{f_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{C}(K) \), where \( f_\alpha(x) = \frac{x_\alpha}{n} \) for every \( n \in \mathbb{N} \), \( \alpha \in \Gamma_n \) and \( x \in K \).
- Then, \( \mathcal{F} \) separates the points of \( K \) and it does not contain an independent sequence of functions.
The continuous image of a WRN compact space is QWRN.

Zero-dimensional QWRN compacta are WRN.

There exists a zero-dimensional RN compact space $\mathbb{L}_0$ and a continuous surjection $\pi: \mathbb{L}_0 \rightarrow \mathbb{L}_1$ such that $\mathbb{L}_1$ is not RN.
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Zero-dimensional QWRN compacta are WRN.

Theorem (A. Avilés and P. Koszmider, 2011)

There exists a zero-dimensional RN compact space $\mathbb{L}_0$ and a continuous surjection $\pi: \mathbb{L}_0 \rightarrow \mathbb{L}_1$ such that $\mathbb{L}_1$ is not RN.

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Corollary

There is a QRN compact space which is not WRN.
Theorem

Every WRN compact space is in the class MS.
Theorem

Every WRN compact space is in the class MS.

Proof.

- $K$ WRN $\Rightarrow \mathcal{C}(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\text{span} \mathcal{F} = \mathcal{C}(K)$.
Theorem

Every WRN compact space is in the class MS.

Proof.

- $K$ WRN $\Rightarrow \mathcal{C}(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\text{span} \mathcal{F} = \mathcal{C}(K)$.

- If $\mu$ is a Radon measure on $K$, the operator $T : \mathcal{C}(K) \to L^1(\mu)$ which takes every continuous function to its equivalence class in $L^1(\mu)$ is Dunford-Pettis and has dense range.
Theorem

Every WRN compact space is in the class MS.

Proof.

- $K$ WRN $\Rightarrow C(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\overline{\text{span } \mathcal{F}} = C(K)$.

- If $\mu$ is a Radon measure on $K$, the operator $T : C(K) \to L^1(\mu)$ which takes every continuous function to its equivalence class in $L^1(\mu)$ is Dunford-Pettis and has dense range.

- Therefore, $T(\mathcal{F})$ is a relatively $\| \cdot \|$-compact space with $\overline{\text{span } T(\mathcal{F})} = L^1(\mu)$. 
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- $K$ WRN $\Rightarrow \mathcal{C}(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\text{span} \mathcal{F} = \mathcal{C}(K)$.

- If $\mu$ is a Radon measure on $K$, the operator $T : \mathcal{C}(K) \to L^1(\mu)$ which takes every continuous function to its equivalence class in $L^1(\mu)$ is Dunford-Pettis and has dense range.

- Therefore, $T(\mathcal{F})$ is a relatively $\| \cdot \|$-compact space with $\text{span} T(\mathcal{F}) = L^1(\mu)$.

- In particular, $T(\mathcal{F})$ and $L^1(\mu)$ are separable $\Rightarrow \mu$ is separable.
Every linearly ordered compact space is WRN (E. Glasner, M. Megrelishvili)
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If there exists a continuous surjective function $p : K \to [0, 1]^{\omega_1}$, then $K$ is not WRN.
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**Theorem (Stegall/J. Orihuela-W. Schachermeyer-M. Valdivia, 1991)**

*A compact space $K$ is Eberlein if and only if it is Corson and RN.*
Weakly Radon-Nikodým compact spaces
Stability under continuous images
Existence of convergent sequences

QRN $\xrightarrow{\text{Eberlein}}$ QWRN
$\uparrow$ $\uparrow$
RN $\xrightarrow{\text{Corson}}$ WRN $\xrightarrow{\text{MS}}$ MS
Theorem (V. Farmaki, 1985)

A compact space \( K \subset \Sigma(\Gamma) \) is Eberlein if and only if for every \( \varepsilon > 0 \) there exists a decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n^\varepsilon \) such that for every \( x \in K \) and every \( n \in \mathbb{N} \), the set \( \{ \gamma \in \Gamma_n^\varepsilon : |x_\gamma| > \varepsilon \} \) is finite.
Theorem (V. Farmaki, 1985)

A compact space $K \subset \Sigma(\Gamma)$ is Eberlein if and only if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma^\varepsilon_n$ such that for every $x \in K$ and every $n \in \mathbb{N}$, the set $\{\gamma \in \Gamma^\varepsilon_n : |x_\gamma| > \varepsilon\}$ is finite.

The Talagrand’s compact $T \subset \{0, 1\}^{\omega^\omega}$ consisting of all functions $1_A$ with $A \subset \omega^\omega$ for which there exist $n \in \mathbb{N}$ and $s \in \omega^n$ such that

$$x|_n = y|_n = s \text{ but } x|_{n+1} \neq y|_{n+1} \text{ for every } x, y \in A \text{ with } x \neq y$$

is an example of a Corson compact that is not Eberlein.
Theorem (A. D. Arvanitakis, 2002)

A compact space $K$ is Eberlein if and only if it is Corson and QRN.
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A compact space $K$ is Eberlein if and only if it is Corson and QRN.

Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x1_A \in K$. 

Gonzalo Martínez Cervantes
Theorem (A. D. Arvanitakis, 2002)

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Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x1_A \in K$. Then, $K$ is QWRN if and only if it is Eberlein.
Theorem (A. D. Arvanitakis, 2002)

A compact space $K$ is Eberlein if and only if it is Corson and QRN.

Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x1_A \in K$. Then, $K$ is QWRN if and only if it is Eberlein.

Corollary

The Talagrand’s compact is solid, Corson and not Eberlein. Therefore, the Talagrand’s compact is not QWRN.
Theorem (A. D. Arvanitakis, 2002)

A compact space $K$ is Eberlein if and only if it is Corson and QRN.

Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x1_A \in K$. Then, $K$ is QWRN if and only if it is Eberlein.

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Table of Contents

1 Weakly Radon-Nikodým compact spaces

2 Stability under continuous images

3 Existence of convergent sequences
Weakly Radon-Nikodým compact spaces
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\[ \text{QRN} \quad \rightarrow \quad \text{QWRN} \]
\[ \uparrow \quad \uparrow \]
\[ \text{RN} \quad \rightarrow \quad \text{WRN} \quad \rightarrow \quad \text{MS} \]

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There exists a WRN compact space which is not sequentially compact.

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Proof.

Let \( \mathcal{R} \) be a maximal family of subsets of \( \mathbb{N} \) with respect to the condition that for \( R, S \in \mathcal{R} \) at least one of the sets \( R \cap S \), \( R \cap S^c \) or \( R^c \cap S \) is finite.

There exists a WRN compact space which is not sequentially compact.

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- The set $C = \{1_R : R \in \mathcal{R}\}$ of $\ell_\infty$ is weakly precompact.
- Take $X = \text{span} \ C \subset \ell_\infty$. 

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- The set $C = \{1_R : R \in \mathcal{R}\}$ of $\ell_\infty$ is weakly precompact.

- Take $X = \text{span} C \subset \ell_\infty$. Since $X$ is WPG, $K = B_{X^*}$ is WRN.

There exists a WRN compact space which is not sequentially compact.

Proof.

- Let $\mathcal{R}$ be a maximal family of subsets of $\mathbb{N}$ with respect to the condition that for $R, S \in \mathcal{R}$ at least one of the sets $R \cap S$, $R \cap S^c$ or $R^c \cap S$ is finite.
- The set $C = \{1_R : R \in \mathcal{R}\}$ of $\ell_\infty$ is weakly precompact.
- Take $X = \text{span} \ C \subset \ell_\infty$. Since $X$ is WPG, $K = B_{X^*}$ is WRN.
- The sequence $(e_n^*)_{n=1}^\infty$ does not have convergent subsequences in $K$. 

\[\square\]
Question (R. Haydon, 1980)

*Does every WRN compact space contain a nontrivial convergent sequence?*
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*Does every infinite compact space contain either a nontrivial convergent sequence or else a copy of $\beta\mathbb{N}$?*
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- A compact space is said to be an Efimov space if it is a counterexample to Efimov’s Problem.
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- A compact space is said to be an Efimov space if it is a counterexample to Efimov’s Problem.
- Efimov spaces exist under various set-theoretic assumptions.
- It is unknown whether a positive answer is consistent with ZFC.
Set an ordinal $\epsilon > 0$. An inverse system is a family $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ of compact spaces $K_\alpha$ and continuous functions $f_{\alpha,\beta} : K_\beta \to K_\alpha$ such that $f_{\alpha,\gamma} \circ f_{\gamma,\beta} = f_{\alpha,\beta}$ for any $\alpha < \gamma < \beta$. 
Definition

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- If $\epsilon$ is a limit ordinal, then the **limit of the inverse system** is the subspace of $\prod_{\alpha<\epsilon} K_\alpha$ consisting of those points $(x_\alpha)_{\alpha<\epsilon}$ that satisfy $f_{\alpha,\beta}(x_\beta) = x_\alpha$ for every $\alpha < \beta$.
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$\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ is said to be continuous if for every limit ordinal $\gamma < \epsilon$, $K_\gamma$ is the limit of the inverse system $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \gamma \rangle$. 
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- Set an ordinal $\epsilon > 0$. An inverse system is a family $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ of compact spaces $K_\alpha$ and continuous functions $f_{\alpha,\beta} : K_\beta \to K_\alpha$ such that $f_{\alpha,\gamma} \circ f_{\gamma,\beta} = f_{\alpha,\beta}$ for any $\alpha < \gamma < \beta$.

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- $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ is said to be continuous if for every limit ordinal $\gamma < \epsilon$, $K_\gamma$ is the limit of the inverse system $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \gamma \rangle$.

- $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ is said to be based on simple extensions if for every $\alpha < \epsilon$ there exists a point $x_\alpha \in K_\alpha$ such that $|f_{\alpha,\alpha+1}^{-1}(x)| = 1$ if $x \neq x_\alpha$ and $|f_{\alpha,\alpha+1}^{-1}(x_\alpha)| = 2$. 
Theorem (Koppelberg)

If $\langle f_{\alpha, \beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ is a continuous inverse system based on simple extensions with limit $K$, then $K$ does not map onto $[0, 1]^{\omega_1}$ unless $K_0$ does.
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Theorem (M. Džamonja and G. Plebanek, 2007)

Let $\langle f_{\alpha, \beta}, K_\alpha : \alpha < \beta < \omega_1 \rangle$ be a continuous inverse system based on simple extensions with limit $K$. If $K_0 = 2^\omega$, then $K$ is in the class MS.
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Question

Let \( \langle f_{\alpha, \beta}, K_{\alpha} : \alpha < \beta < \omega_1 \rangle \) be a continuous inverse system based on simple extensions with \( K_0 = 2^\omega \) and with limit \( K \).
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Question

Let \( \langle f_{\alpha, \beta}, K_{\alpha} : \alpha < \beta < \omega_1 \rangle \) be a continuous inverse system based on simple extensions with \( K_0 = 2^{\omega} \) and with limit \( K \). Is \( K \) WRN?
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