Some structural properties of ideal invariant injections

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joint work with Marek Balcerzak and Szymon Głąb
We will work with injections from $\omega$ to $\omega$. The set of all such injections will be denoted by $\text{Inj}$. Fix an ideal $\mathcal{I}$ on $\omega$ and let $f \in \text{Inj}$. We say that $f$ is $\mathcal{I}$-invariant if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that $f^{-1}$ is $\mathcal{I}$-invariant if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If $f$ and $f^{-1}$ are $\mathcal{I}$-invariant then $f$ is called $\text{bi-}\mathcal{I}$-invariant. Note that every $f \in \text{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

**Fact**

Let $\mathcal{I}$ be an ideal on $\omega$ and let $f \in \text{Inj}$.

(i) $f^{-1}$ is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then $f$ is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\text{Fix}(f) \in \mathcal{I}^*$, then $f$ is bi-$\mathcal{I}$-invariant.

(iv) $\text{Inj}$ is a $G_\delta$ subset of $\omega^\omega$, hence it is a Polish space.
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(iv) \( \text{Inj} \) is a \( G_\delta \) subset of \( \omega^\omega \), hence it is a Polish space.
Easy examples

(i) Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$, hence $f$ is not bi-$\mathcal{I}_d$-invariant.

(ii) Let $f : \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n + 1) = 4n + 2$, $f(4n + 3) := 2n + 1$ for $n \in \omega$. Then $f$ is a bijection. Consider the ideal $\mathcal{I}$ defined as follows

$$\mathcal{I} := \{A \cup B : A \in \text{Fin}, B \subseteq 2\omega\}.$$ 

Clearly, $f$ is $\mathcal{I}$-invariant bijection which is not bi-$\mathcal{I}$-invariant.
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Clearly, $f$ is $\mathcal{I}$-invariant bijection which is not bi-$\mathcal{I}$-invariant.
Fact

There are three types of countably generated ideals: \( \text{Fin} \), \( \text{Fin} \oplus \mathcal{P}(\omega) \) and \( \text{Fin} \times \emptyset \).

Theorem

- Each \( f \in \text{Inj} \) is bi-Fin-invariant.
- Sets \( \text{Fin} \oplus \mathcal{P}(\omega)-\text{Inv} \), of all \( \text{Fin} \oplus \mathcal{P}(\omega) \)-invariant injections, and \( \text{bi-Fin} \oplus \mathcal{P}(\omega)-\text{Inv} \), of all bi-Fin \( \oplus \mathcal{P}(\omega) \)-invariant injections, are true \( F_\sigma \) subsets of \( \text{Inj} \).
- The sets \( \text{Fin} \times \emptyset-\text{Inv} \) and \( \text{bi-Fin} \times \emptyset-\text{Inv} \), are meager of type \( F_{\sigma\delta} \) in \( \text{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega} \). Moreover, \( \text{bi-I-Inv} \) is \( F_{\sigma\delta} \)-complete.
Countably generated ideals

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Fact

There are three types of countably generated ideals: Fin, Fin $\oplus \mathcal{P}(\omega)$ and Fin $\times \emptyset$.

Theorem

- Each $f \in \text{Inj}$ is bi-Fin-invariant.
- Sets Fin $\oplus \mathcal{P}(\omega)$-$\text{Inv}$, of all Fin $\oplus \mathcal{P}(\omega)$-invariant injections, and bi-Fin $\oplus \mathcal{P}(\omega)$-$\text{Inv}$, of all bi-Fin $\oplus \mathcal{P}(\omega)$-invariant injections, are true $F_\sigma$ subsets of Inj.
- The sets Fin $\times \emptyset$-$\text{Inv}$ and bi-Fin $\times \emptyset$-$\text{Inv}$, are meager of type $F_{\sigma\delta}$ in Inj $\subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, bi-$\mathcal{I}$-$\text{Inv}$ is $F_{\sigma\delta}$-complete.
Fact

There are three types of countably generated ideals: Fin, Fin $\oplus \mathcal{P}(\omega)$ and Fin $\times \emptyset$.

Theorem

- Each $f \in \text{Inj}$ is bi-Fin-invariant.
- Sets Fin $\oplus \mathcal{P}(\omega)$-$\text{Inv}$, of all Fin $\oplus \mathcal{P}(\omega)$-invariant injections, and bi-Fin $\oplus \mathcal{P}(\omega)$-$\text{Inv}$, of all bi-Fin $\oplus \mathcal{P}(\omega)$-invariant injections, are true $F_\sigma$ subsets of $\text{Inj}$.
- The sets Fin $\times \emptyset$-$\text{Inv}$ and bi-Fin $\times \emptyset$-$\text{Inv}$, are meager of type $F_{\sigma\delta}$ in $\text{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, bi-$\mathcal{I}$-$\text{Inv}$ is $F_{\sigma\delta}$-complete.
Maximal ideals

**I-invariance**

Let $\mathcal{I}$ be a maximal ideal. Then $f \in \text{Inj}$ is $\mathcal{I}$-invariant iff $f[\omega] \in \mathcal{I}$ or $\text{Fix}(f) \in \mathcal{I}^*$. 

EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

**Corollary**

Let $\mathcal{I}$ be a maximal ideal on $\omega$ and $f \in \text{Inj}$. Then $f$ is $\mathcal{I}$-invariant if and only if either $\text{Fix}(f) \in \mathcal{I}^*$ or $f[\omega] \in \mathcal{I}$.

**Example**

Let $\mathcal{I}, \mathcal{J}$ be non-isomorphic maximal ideals on $\omega$ and $f \in \text{Inj}$. Then $f$ is bi-$\mathcal{I} \oplus \mathcal{J}$-invariant iff $\text{Fix}(f) \in (\mathcal{I} \oplus \mathcal{J})^*$.
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Maximal ideals

$I$-invariance

Let $I$ be a maximal ideal. Then $f \in \text{Inj}$ is $I$-invariant iff $f[\omega] \in I$ or $\text{Fix}(f) \in I^*$.  
EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

Corollary

Let $I$ be a maximal ideal on $\omega$ and $f \in \text{Inj}$. Then $f$ is $I$-invariant if and only if either $\text{Fix}(f) \in I^*$ or $f[\omega] \in I$.

Example

Let $I, J$ be non-isomorphic maximal ideals on $\omega$ and $f \in \text{Inj}$. Then $f$ is bi-$I \oplus J$-invariant iff $\text{Fix}(f) \in (I \oplus J)^*$.
Maximal ideals

\(\mathcal{I}\)-invariance

Let \(\mathcal{I}\) be a maximal ideal. Then \(f \in \text{Inj}\) is \(\mathcal{I}\)-invariant iff \(f[\omega] \in \mathcal{I}\) or \(\text{Fix}(f) \in \mathcal{I}^*\).

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Corollary

Let \(\mathcal{I}\) be a maximal ideal on \(\omega\) and \(f \in \text{Inj}\). Then \(f\) is \(\mathcal{I}\)-invariant if and only if either \(\text{Fix}(f) \in \mathcal{I}^*\) or \(f[\omega] \in \mathcal{I}\).

Example

Let \(\mathcal{I}, \mathcal{J}\) be non-isomorphic maximal ideals on \(\omega\) and \(f \in \text{Inj}\). Then \(f\) is bi-\(\mathcal{I} \oplus \mathcal{J}\)-invariant iff \(\text{Fix}(f) \in (\mathcal{I} \oplus \mathcal{J})^*\).
A submeasure on $\omega$ is a function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$;
- if $A \subset B$ then $\varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \omega$.

A submeasure $\varphi$ is called a lower semicontinuous submeasure (in short, lscsm) if $\varphi(A) = \lim_{n \to \infty} \varphi(A \cap n)$ for all $A \subset \omega$. For any lscsm $\varphi$, we consider two ideals given by

$$Exh(\varphi) = \{A \subset \omega : \lim_{n \to \infty} \varphi(A \setminus n) = 0\}.$$  

$$Fin(\varphi) = \{A \subset \omega : \varphi(A) < \infty\}.$$
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$$Fin(\varphi) = \{A \subset \omega: \varphi(A) < \infty\}.$$
Let $\varphi$ be a lscsm. Then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, $Fin(\varphi)$ is an $F_{\sigma}$ ideal and $Exh(\varphi) \subset Fin(\varphi)$.

**Theorem [Mazur, Solecki]**

Let $\mathcal{I}$ be an ideal on $\omega$. Then

- $\mathcal{I}$ is an $F_{\sigma}$ ideal if and only if $\mathcal{I} = Fin(\varphi)$ for some lscsm $\varphi$.
- $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = Exh(\varphi)$ for some lscsm $\varphi$.
- $\mathcal{I}$ is an $F_{\sigma\delta}$ P-ideal if and only if $\mathcal{I} = Fin(\varphi) = Exh(\varphi)$ for some lscsm $\varphi$.

If $\mathcal{I}$ is ideal on $\omega$ then it is not a $G_\delta$ set.
Let $\varphi$ be a lscsm. Then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, $Fin(\varphi)$ is an $F_{\sigma}$ ideal and $Exh(\varphi) \subset Fin(\varphi)$.

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Theorem

Let $\varphi$ be a lsc submeasure on $\omega$. Let $f : \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on $f$ such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then $f$ is invariant with respect to the ideals $\text{Fin}(\varphi)$ and $\text{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then $f$ is bi-invariant with respect to the ideals $\text{Fin}(\varphi)$ and $\text{Exh}(\varphi)$.

Remark

by the lower semicontinuity of $\varphi$, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any $A$ with $|A| \leq n$ for some fixed $n$. The answer is ”no”.

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Ideals generated by Solecki’s submeasures

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by the lower semicontinuity of \( \varphi \), one can assume that the condition \( \varphi(A) \geq C_f \varphi(f[A]) \) holds only for finite sets \( A \subseteq \omega \). It is natural to ask whether one can assume that the condition \( \varphi(A) \geq C_f \varphi(f[A]) \) holds for any \( A \) with \( |A| \leq n \) for some fixed \( n \). The answer is ”no”. 
Ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$

Classical density ideal

$$\mathcal{I}_d := \{ A \subset \omega : \frac{\text{card}(A \cap n)}{n} \to 0 \}$$

Classical summable ideal

$$\mathcal{I}_S := \{ A \subset \omega : \sum_{n \in A} \frac{1}{n} < \infty \}$$
Theorem

Let $f : \omega \rightarrow \omega$ be an increasing injection. The following conditions are equivalent:

(i) $f$ is bi-$\mathcal{I}_d$-invariant;
(ii) $d(f[\omega]) > 0$;
(iii) there is $C \in \omega$ such that $f(n) \leq Cn$ for every $n \geq 1$;
(iv) $f$ is bi-$\mathcal{I}_{(1/n)}$-invariant.

Lower density

$$d(A) = \lim \inf \frac{\text{card}(A \cap n)}{n}$$


Thank you for your attention!