Nonseparable growth of $\omega$
supporting a strictly positive measure

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Definitions

**Growth**

A compact space $K$ is a **growth** of $\omega$ if there exists a compactification $\gamma\omega$ of $\omega$ such that $K \simeq \gamma\omega \setminus \omega$. 

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**Strictly positive measure**

A measure $\mu$ on a topological space $X$ is **strictly positive** if $\mu(U) > 0$ for any open $U \subseteq X$.

A finitely additive measure $\mu$ on a Boolean algebra $\mathcal{A}$ is **strictly positive** if $\mu(a) > 0$ for any $a \in \mathcal{A}^+$. 

**Remark**

There is a strictly positive measure on a Boolean algebra $\mathcal{A}$ iff there is a strictly positive measure on the Stone space $\text{ult}(\mathcal{A})$. 


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Trivial examples

Any separable compact space is a growth of $\omega$.
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**Fact**
A Boolean algebra $\mathcal{A}$ can be embedded in $\mathcal{P}(\omega)/\text{fin}$ iff $\text{ult}(\mathcal{A})$ is a growth of $\omega$.
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**Lebesgue measure algebra**

Let $\mathcal{B} = \text{Bor}[0,1]/\mathcal{N}$, where $\mathcal{N} = \{A \subseteq [0,1] : \lambda(A) = 0\}$. It has nonseparable $\text{ult}(\mathcal{B})$ and the measure $\lambda$ transfers to a strictly positive measure on $\text{ult}(\mathcal{B})$. 
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It has nonseparable $\text{ult}(\mathcal{B})$ and the measure $\lambda$ transfers to a strictly positive measure on $\text{ult}(\mathcal{B})$. Assuming CH, by Parovičenko $\text{ult}(\mathcal{B})$ embeds into $\mathcal{P}(\omega)/\text{fin}$, so it is a growth of $\omega$. 
However...

**Dow & Hart:** Under Open Coloring Axiom the measure algebra does not embed into $\mathcal{P}(\omega)/\text{fin}$. 
Problem

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Question

Is there a ZFC example of nonseparable growth of $\omega$ which supports a strictly positive measure?
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**Question**

Is there a ZFC example of nonseparable growth of $\omega$ which supports a strictly positive measure? Equivalently: is there a ZFC example of a Boolean algebra with nonseparable Stone space that supports a strictly positive measure and can be embedded into $\mathcal{P}(\omega)/\text{fin}$?
Bell, van Mill and Todorčević: ZFC examples of compactifications of $\omega$ with nonseparable ccc remainders
Related results

- **Bell, van Mill and Todorčević:** ZFC examples of compactifications of $\omega$ with nonseparable ccc remainders

- **Drygier & Plebanek:** under $b = c$ (or some weaker statement) an example of $\gamma\omega$ with nonseparable remainder supporting a strictly positive measure
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Borodulin-Nadzieja & Inamdar: ZFC example of nonseparable growth of $\omega$ supporting a strictly positive measure
Asymptotic density

$$d(A) = \lim_{n \to \infty} \frac{|\{m < n : m \in A\}|}{n},$$

if the limit exists for $A \subseteq \omega$. As $d(A) = 0$ for finite $A$, we can define also asymptotic density on $\mathcal{P}(\omega)/\text{fin}$. 
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Measure algebra, continued

**Frankiewicz & Gutek:** Under CH, there is an embedding \( \Phi : \mathcal{B} \to \mathcal{P}(\omega)/\text{fin} \) such that \( \lambda(b) = d(\Phi(b)) \) for any \( b \in \mathcal{B} \).
Main result

**Theorem**

There exists a Boolean algebra $\mathcal{A}$ with the following properties:

- $\text{ult}(\mathcal{A})$ is not separable
- there exists a strictly positive measure $\mu$ on $\mathcal{A}$
- there exists an embedding $\Psi : \mathcal{A} \rightarrow \mathcal{P}(\omega)/\text{fin}$ such that $\mu(a) = d(\Psi(a))$ for any $a \in \mathcal{A}$.
### Notations

\[
\{ P_\alpha : \alpha < c \} = [2^\omega]^{\leq \omega} \\
\{ B_\alpha : \alpha < c \} - \text{an almost disjoint family in } \mathcal{P}(\omega).
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Construction

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**Definition of generators**

\[ P_\alpha = \{ t_\alpha^n : n \in \omega \} \subseteq 2^\omega \]
\[ B_\alpha = \{ m_\alpha^i : i \in \omega \} \subseteq \omega \]
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\[ \varphi^\alpha_0 = t^\alpha_0|_{\{m^\alpha_0, m^\alpha_1\}}, \quad \varphi^\alpha_1 = t^\alpha_1|_{\{m^\alpha_2, m^\alpha_3, m^\alpha_4\}}, \quad \varphi^\alpha_2 = t^\alpha_2|_{\{m^\alpha_5, m^\alpha_6, m^\alpha_7, m^\alpha_8\}} , \text{ etc.} \]
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We define \( U_\alpha = \bigcup_{i \in \omega} [\varphi_i^\alpha] \).
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**Definition of algebra**

\[ \mathcal{A} = \text{alg} \left( \text{Clop}(2^\omega) \cup \{ U_\alpha : \alpha < c \} \right) \]
Sketch of the proof of main result

- \( \lambda \) is a strictly positive measure on \( \mathcal{A} \)
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- We define for any $\alpha < c$ such $\Psi_0(U_\alpha)$ that $\lambda(U_\alpha) = d(\Psi_0(U_\alpha))$ and we can extend $\Psi_0$ to a homomorphism $\Psi : \mathcal{A} \to \mathcal{P}(\omega)/\text{fin}$
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- $\Psi : \mathcal{A} \to \mathcal{P}(\omega)/\text{fin}$ also transfers the Lebesgue measure to the asymptotic density
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- $\Psi : \mathcal{A} \to \mathcal{P}(\omega)/\text{fin}$ also transfers the Lebesgue measure to the asymptotic density
- The homomorphism $\Psi$ is an embedding, which is an easy corollary from transferring the measure to density
Theorem (Borodulin-Nadzieja, Inamdar, 2015)

There is a Boolean algebra $\mathcal{I} \subseteq \mathcal{P}(\omega)/\text{Fin}$ such that

- $\mathcal{I}$ is not $\sigma$-centered,
- $\mathcal{I} \Vdash \mathcal{B} \text{ "}\mathcal{I} \text{ is } \sigma\text{-centered".}$
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Theorem (Kamburelis, 80’)

If $\mathcal{C}$ is a Boolean algebra and $\vdash_{\mathcal{B}} \text{"}\mathcal{B} \text{ is } \sigma \text{-centered"}$, then $\mathcal{C}$ supports a strictly positive measure.