Combinatorics related to the Michael space problem

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Definition

An ultrafilter $\mathcal{U}$ is a *Michael Ultrafilter* if for every compact set $K \subseteq \omega^\omega$ if $\mathfrak{d}_\mathcal{U}(K) > \omega$, then $\mathfrak{d}_\mathcal{U}(K) \geq \mathfrak{d}_\mathcal{U}$.

Definition

A topological space X is Lindelöf if every open cover has an countable open subcover.

E. Michael, 195X

Is there a Lindelöf space X such that $X \times \omega^\omega$ is not Lindelöf?

Such spaces are called *Michael spaces*.

Examples of Michael Spaces:
Under CH ($\mathfrak{b} = \omega_1$) there is one (E. Michael).
Under $\mathfrak{d} = \text{cov}(\mathcal{M})^*$ there is one (J. Moore).

Theorem (Moore)

($\mathfrak{d} < \aleph_\omega$). There is a Michael space if and only if there exists is a sequence $\{X_\alpha : \alpha \in \kappa\}$ with $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$ such that

- *$(X_\alpha)_{\alpha \leq \kappa}$ is $\subsetneq$-increasing and $X_\kappa = \omega^\omega$*
- *For each compact set $K \subseteq \omega^\omega$, the least ordinal α_K such that $K \subseteq X_{\alpha_K}$ has at most countable cofinality.*

$\text{cov}(\mathcal{M})$ is the smallest number of meager sets whose union covers the real line.

A Michael sequence

Example: Suppose $\mathfrak{b} = \omega_1$ and let $\{f_\alpha : \alpha \in \omega_1\}$ be an unbounded family. If $X_\alpha = \{f \in \omega^\omega : f \not\geq^* f_\gamma \text{ for } \gamma < \alpha\}$, then $\{X_\alpha : \alpha \leq \omega_1\}$ is a Michael sequence.

Another Michael sequence

Example: Suppose $\mathfrak{d} = \text{cov}(\mathcal{M})$ and let $\{f_\alpha : \alpha \in \mathfrak{d}\}$ be a dominating family. If $X_\alpha = \{f \in \omega^\omega : f \leq f_\gamma \text{ for } \gamma < \alpha\}$, then $\{X_\alpha : \alpha \leq \mathfrak{d}\}$ is a Michael sequence.

Definition

Let $\mathcal{U}$ be an ultrafilter over ω. If $f, g \in \omega^\omega$, then $f \leq_{\mathcal{U}} g$ if $\{n \in \omega : g(n) \geq f(n)\} \in \mathcal{U}$.

$\leq_{\mathcal{U}}$ is a total order for ω^ω, therefore being $\leq_{\mathcal{U}}$-unbounded is the same thing as being $\leq_{\mathcal{U}}$-dominating.

$$\mathfrak{d}_{\mathcal{U}} = \text{cof}(\omega^\omega/\mathcal{U}) = \min\{|A| : A \subseteq \omega^\omega \text{ is } \leq_{\mathcal{U}} \text{-dominating}\}.$$

$$\mathfrak{b} \leq \mathfrak{d}_{\mathcal{U}} \leq \mathfrak{d}$$

Cardinal Invariants

$$\mathfrak{b} = \min\{|A| : A \subseteq \omega^\omega \text{ is } < \text{-unbounded}\}.$$

$$\mathfrak{b}^* = \min\{|A| : A \subseteq \omega^\omega \text{ is } < \text{-unbounded everywhere}\}.$$

$$\mathfrak{d} = \min\{|A| : A \subseteq \omega^\omega \text{ is } \leq \text{-dominating}\}.$$

Internal Cardinal Invariants: If $K \subseteq \omega^\omega$

$$\mathfrak{b}(K) = \min\{|A| : A \subseteq K \text{ is } <|_K\text{-unbounded}\}.$$

$$\mathfrak{b}^*(K) = \min\{|A| : A \subseteq \omega^\omega \text{ is } <|_K\text{-unbounded everywhere}\}.$$

$$\mathfrak{d}(K) = \min\{|A| : A \subseteq \omega^\omega \text{ is } \leq|_K\text{-dominating}\}.$$

Definition

An ultrafilter $\mathcal{U}$ is a *Michael Ultrafilter* if for every compact set $K \subseteq \omega^\omega$ if $\mathfrak{d}_\mathcal{U}(K) > \omega$, then $\mathfrak{d}_\mathcal{U}(K) \geq \mathfrak{d}_\mathcal{U}$.

Theorem

If $\mathcal{U}$ is an ultrafilter with $\mathfrak{d}_\mathcal{U} = \omega_1$ then $\mathcal{U}$ is Michael. In particular, under $\mathfrak{d} = \omega_1$, every ultrafilter is a Michael ultrafilter.

Theorem

If $\text{cov}(\mathcal{M}) = \mathfrak{c}$, then there are Michael ultrafilters.

Are there ultrafilter properties that imply the existence of a Michael ultrafilter?

Theorem

On Miller's Model, every P-point is not a Michael ultrafilter

The distributive number $\mathfrak{h}$ is the smallest number of dense open families of subsets of $[\omega]^\omega$ with empty intersection.

Theorem

If $\mathfrak{t} = \mathfrak{h}$, *then* $P(\omega)/\mathrm{FIN} \Vdash$ "$\dot{\mathcal{U}}_{gen}$ *is a Michael ultrafilter*"

Theorem

For every compact set K,
$P(\omega)/\mathrm{FIN} \Vdash$ "$\mathfrak{d}_{\dot{\mathcal{U}}_{gen}}(K) = \min_{U \in \dot{\mathcal{U}}_{gen}} \mathfrak{b}^*(K|_U)$"

Theorem (Hrúšak, Rojas, Zapletal)

There is an F_σ ideal I such that $\mathrm{cof}(\mathrm{I}) = \omega_1$ in the Mathias, Laver and Miller model.

Using this ideal, it is possible to cook (inside those models) a compact set such that $\mathfrak{b}^*(K|_A) = \omega_1$.

Theorem

Being a Ramsey Ultrafilter does not neccesarily imply that the ultrafilter is Michael.

Model	Michael Space	Michael Ultrafilter	After $P(\omega)/\mathrm{FIN}$
Cohen	Yes	Yes	Yes
Random	Yes	Yes	Yes
Sacks	Yes	Yes	Yes
Miller	Yes	?	Yes
Mathias	?	?	?
Laver	?	?	?

Thanks for your attention!!!1