The botanics of provability (and ω^ω other short stories).

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Outline

Topological models of provability logics

joint work with David Fernández-Duque.

Provability logic

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iv. $[[\Box\varphi]] = \mathrm{Pr}[[\varphi]]$.

Provability logic

Theorem (Solovay)

tfae:

1. $\mathrm{GL} \vdash \varphi$,
2. $\mathrm{PA} \vdash [[\varphi]]$ *for any realization* $[[\cdot]]$.

Kripke models

- A *Kripke model* is a directed graph (or frame) (X, R) together with a *valuation*, i.e., a mapping $[[\cdot]]$ from modal formulae to subsets of X such that

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- equivalently, a modal logic L is *complete* with respect to a class of frames $\mathcal{F}$ if any formula consistent with L has a model based on some $X \in \mathcal{F}$.
- a modal logic L is strongly complete with respect to a class of frames $\mathcal{F}$ if any set of formulae consistent with L has a model based on some $X \in \mathcal{F}$.

Kripke models

Theorem (Segerberg)

tfae:

1. $\mathsf{GL} \vdash \varphi$,
2. ϕ *is valid in all transitive, converse well-founded Kripke frames.*

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- Since those are small trees, I'll call them flowers.
- as is well known, GL is not strongly complete with respect to any class of frames.

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$$\Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \rightarrow \Diamond p_{i+1}) \colon i < \omega\}$$

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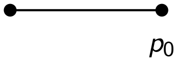

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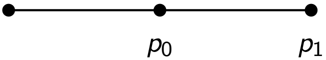

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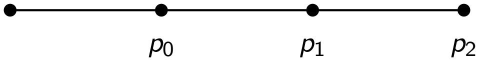

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we write $X \models \varphi$ if $[[\varphi]] = X$ for any realization $[[\cdot]]$.

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- we denote this topology by τ_R.
- It is easy to see that this coincides with the previous interpretation.

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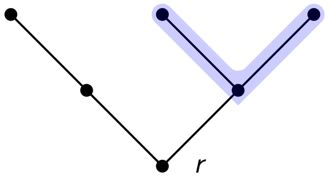

Figure: a flower.

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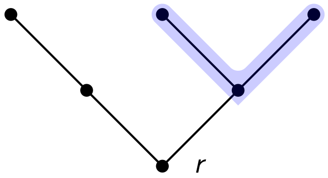

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We say (X, τ) is a scattered space if some $d^{\xi} X$ is empty (alternatively, if any subspace has an isolated point).
If X is scattered and $x \in X$, we call the rank of x the least ordinal ξ such that $x \notin d^{\xi+1}(X)$.

GL again.

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- Lob's axiom is valid in a topological space iff it is scattered.

Completeness theorems

Theorem (Esakia)

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This result can be improved.

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- Therefore, a consistent finite set of formulae can be satisfied on a collection of flowers.

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- Therefore, a consistent finite set of formulae can be satisfied on a *bouquet*.

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- We say a topological space (T, σ) is an ω-bouquet if there exists a binary relation R on T such that (T, R) is a countable, converse well-founded tree and $\sigma = \sigma_R$.

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GL *is strongly complete with respect to the set of all* ω*-bouquets.*

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- This procedure can somehow be iterated transfinitely to yield topologies τ_λ.

even more Completeness

Theorem

Let λ be a nonzero ordinal. If (X, τ) is tall enough, then GL is strongly complete with respect to $(X, \tau_{+\lambda})$.

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Theorem

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- Particular cases of this theorem are as follows:

Example

Let X be an ordinal number. $\mathcal{I}$, the *initial topology*, is generated by all segments $[0, \alpha]$.

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Let X be an ordinal number. Then τ_c, the *club topology*, is generated by $\mathcal{I}_{+1}$ and all $\mathcal{I}_{+1}$-limit sets.

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Example

Let X be an ordinal number. Then τ_c, the *club topology*, is generated by $\mathcal{I}_{+1}$ and all $\mathcal{I}_{+1}$-limit sets.
Alternatively, $U \ni x$ contains a neighborhood of x iff x has countable cofinality or U is a club in x.

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GL *is strongly complete with respect to an ordinal* α *with the topology* $\mathcal{I}_{+1}$ *iff* $\alpha > \omega^\omega$.

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GL *is strongly complete with respect to an ordinal* α *with the topology* τ_{c+1} *iff* $\alpha > \omega_{\omega^\omega+1}$.

Thank you!