Compact sets in Euclidean spaces as IFS-attractors

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joint work with T. Banakh
Which sets can be embedd in Euclidean space?

Definition

A metric space \((X, d)\) is called **doubling** if there exists a natural number \(M\) such that each open ball \(B(x, r)\) is contained in the union of at most \(M\) open balls \(B(y, \frac{r}{2})\).
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**Assouad’s theorem**
For each doubling space $(X, d)$ and for each $\alpha \in (0, 1)$ there exists $n \in \mathbb{N}$ and bi-Lipschitz function $\varphi : (X, d^\alpha) \to \mathbb{R}^n$. 

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**Definition**

An **Iterated Function System** (IFS) is a finite collection of contractions on the metric space $X$:

$$\mathcal{F} = \{ f_1, f_2, \ldots, f_n : X \to X ; \max_{i=1,\ldots,n} \{ \text{Lip} f_i \} < 1 \}.$$ 

A nonempty compact set $A \subset X$ which is invariant by the IFS $\mathcal{F}$, in the sense:

$$A = f_1(A) \cup f_2(A) \cup \cdots \cup f_n(A)$$

is called the **attractor of the IFS** $\mathcal{F}$ (IFS-attractor).
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Theorem

For every IFS on a complete metric space $X$ there exist a unique IFS-attractor.
Being an IFS-attractor is not a topological invariant

Problem
Which compact space is homeomorphic to an IFS-attractor in the Euclidean space?

Definition
A compact space $A$ is a Euclidean fractal if it is homeomorphic to some IFS-attractor in $\mathbb{R}^n$ (there exists a metric on $A$ and IFS $F = \{f: A \to A\}$ such that $A = \bigcup_{f \in F} f(A)$).
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Fact

Bi-Lipschitz image of IFS-attractor is also IFS-attractor.
Being an IFS-attractor is a bi-Lipschitz invariant

**Fact**

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\begin{align*}
X \xrightarrow{\mathcal{F}} X \\
\varphi^{-1} \downarrow \quad \varphi \\
\varphi(X) \xrightarrow{\mathcal{F}'} \varphi(X)
\end{align*}
\]

\(X\) - IFS-attractor for family \(\mathcal{F}\) and each \(f \in \mathcal{F}\) is \(\lambda\)-Lipschitz in \(X\) \((\lambda < 1)\).
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**Fact**

Bi-Lipschitz image of IFS-attractor is also IFS-attractor.

\[ X \xrightarrow{\mathcal{F}^k} X \]

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\[ X \text{ - IFS-attractor for family } \mathcal{F} \text{ and each } f \in \mathcal{F} \text{ is } \lambda\text{-Lipschitz in } X \text{ (} \lambda < 1\text{).} \]

For every \( k \in \mathbb{N} \), \( X \) is an IFS-attractor for the family \( \mathcal{F}^k = \{ f_1 \circ \cdots \circ f_k : f_1, \ldots, f_k \in \mathcal{F} \} \) of a \( \lambda^k \)-Lipschitz function.
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**Fact**

Bi-Lipschitz image of IFS-attractor is also IFS-attractor.

\[ X \xrightarrow{F^k} X \quad \xrightarrow{\phi^{-1}} \quad X \xrightarrow{\phi} X \]

\[ \phi(X) \xrightarrow{F'} \phi(X) \]

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\( \phi \) - bi-Lipschitz function so \( \phi : X \to \phi(X) \) is a homeomorphism and \( \phi, \phi^{-1} \) are Lipschitz.
Being an IFS-attractor is a bi-Lipschitz invariant

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is \(\lambda\)-Lipschitz in \(X\) \((\lambda < 1)\).

For every \(k \in \mathbb{N}\), \(X\) is an IFS-attractor for the family \(\mathcal{F}^k = \{f_1 \circ \cdots \circ f_k : f_1, \ldots, f_k \in \mathcal{F}\}\) of a \(\lambda^k\)-Lipschitz function.

\(\varphi\) - bi-Lipschitz function so \(\varphi : X \to \varphi(X)\) is a homeomorphism and \(\varphi, \varphi^{-1}\) are Lipschitz.

Take a \(k \in \mathbb{N}\) such that \(\text{Lip} \varphi \cdot \lambda^k \cdot \text{Lip} \varphi^{-1} < 1\) then \(\varphi(X)\) is an IFS-attractor for \(\mathcal{F}' = \{\varphi \circ f_1 \circ \cdots \circ f_k \circ \varphi^{-1} : f_1, \ldots, f_k \in \mathcal{F}\}\).
Corollary

Each IFS-attractor which is doubling, is an Euclidean fractal.
A sufficient condition of being Euclidean fractal

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Each IFS-attractor which is doubling, is an Euclidean fractal.

\[(X, d) \quad (X, d) \text{ is an IFS-attractor for } \lambda\text{-Lipschitz function from } F (\lambda < 1).\]

\[(X, d^\alpha) \downarrow \text{homeo.} \downarrow
(\varphi \quad \text{bi-Lipschitz})
\]

\[\mathbb{R}^n\]
A sufficient condition of being Euclidean fractal

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Each IFS-attractor which is doubling, is an Euclidean fractal.

$$(X, d) \xrightarrow{h \text{ homeo.}} (X, d')$$

$(X, d)$ is an IFS-attractor for $\lambda$-Lipschitz function from $\mathcal{F}$ ($\lambda < 1$).

$$(X, d^\alpha) \xrightarrow{\varphi \text{ bi-Lipschitz}} \mathbb{R}^n$$

$(X, d^\alpha)$ is an IFS-attractor for $\lambda^\alpha$-Lipschitz functions from $\mathcal{F}$ ($\alpha \in (0, 1)$).
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Each IFS-attractor which is doubling, is an Euclidean fractal.

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\downarrow h \quad \downarrow \quad \downarrow \quad \downarrow \\
(X, d^\alpha) & \quad (X, d^\alpha) \text{ is an IFS-attractor for } \\
\downarrow \quad \downarrow \quad \varphi \quad \downarrow \quad \varphi \quad \downarrow \quad \downarrow \\
\mathbb{R}^n & \quad \varphi(X) \subset \mathbb{R}^n \text{ is an IFS-attractor.}
\end{align*}
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Theorem (Banakh, N 2015)

Let $X$ be compact doubling space and $Z$ be its uncountable, zero-dimensional, subset open in $X$. Then $X$ is an Euclidean fractal.
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1992 - Duvall & Husch ($X \subset \mathbb{R}^n$ and $Z$ - Cantor set)
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Cantor set

compact set

homeo

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$Z$ zero-dim uncountable

compact set

Lipschitz

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