Strong Homology

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Theorem (Eilenberg Steenrod 1945)

Exactly one homology theory $H_*$ satisfies the Eilenberg-Steenrod axioms on the category of finite CW-complexes.
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Exactly one homology theory $H_\ast$ satisfies the Eilenberg-Steenrod axioms on the category of finite CW-complexes.

**Formal definition**

The Eilenberg–Steenrod axioms apply to a sequence of functors $H_n$ from the category of pairs $(X, A)$ of topological spaces to the category of abelian groups, together with a natural transformation $\partial : H_i(X, A) \to H_{i-1}(A)$ called the **boundary map** (here $H_{i-1}(A)$ is a shorthand for $H_{i-1}(A, \emptyset)$). The axioms are:

1. **Homotopy**: Homotopic maps induce the same map in homology. That is, if $g : (X, A) \to (Y, B)$ is homotopic to $h : (X, A) \to (Y, B)$, then their induced maps are the same.
2. **Excision**: If $(X, A)$ is a pair and $U$ is a subset of $X$ such that the closure of $U$ is contained in the interior of $A$, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism in homology.
3. **Dimension**: Let $P$ be the one-point space; then $H_n(P) = 0$ for all $n \neq 0$.
4. **Additivity**: If $X = \bigsqcup X_\alpha$, the disjoint union of a family of topological spaces $X_\alpha$, then $H_n(X) \cong \bigoplus H_n(X_\alpha)$.
5. **Exactness**: Each pair $(X, A)$ induces a long exact sequence in homology, via the inclusions $i : A \to X$ and $j : X \to (X, A)$:

$$\cdots \to H_n(A) \xrightarrow{i^*} H_n(X) \xrightarrow{j^*} H_n(X, A) \xrightarrow{\partial^*} H_{n-1}(A) \to \cdots$$

If $P$ is the one point space then $H_0(P)$ is called the **coefficient group**. For example, singular homology (taken with integer coefficients, as is most common) has as coefficients the integers.
A homology theory $H_*$ is additive if for every $n$ and every

$$X = \bigsqcup_{\alpha \in A} X_{\alpha}$$

the inclusion maps $i_{\alpha} : X_{\alpha} \to X$ induce an isomorphism

$$i_* : \bigoplus_{\alpha \in A} H_n(X_{\alpha}) \to H_n(\bigsqcup_{\alpha \in A} X_{\alpha})$$
Additivity

Definition

A homology theory $H_\ast$ is *additive* if for every $n$ and every

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the inclusion maps $i_\alpha : X_\alpha \to X$ induce an isomorphism

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Theorem (Milnor 1960)

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Henceforth let $H_*$ denote this (unique) extension to CW.
A number of extensions of $H_\ast$ in turn have been proposed.

Prominent among these is strong homology $\overline{H}_\ast$:

1. $\overline{H}_\ast$ equals $H_\ast$ on CW.
2. $\overline{H}_\ast$ satisfies the Eilenberg-Steenrod axioms on paracompact pairs $(X, A)$.
3. $\overline{H}_\ast$ is a Steenrod-type homology theory.
4. $\overline{H}_\ast$ is strong-shape-invariant.

It was against this background that Sibe Mardešić and Andrei Prasolov asked Is strong homology additive? (1) and (2) imply $\overline{H}_\ast$ additive on CW, and for finite sums, respectively.

So Mardešić and Prasolov began by considering the strong homology of an infinite countable sum $Y$ of Hawaiian earrings $X$.

$\overline{H}_2(X) = 0$.

Mardešić and Prasolov directly computed, though, that $\overline{H}_2(X)$ is the quotient of coherent families by trivial families; $\overline{H}_2(X) = 0$, in other words, iff every coherent family is trivial.

But this, as they and others would show, is a question independent of ZFC.
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It was against this background that Sibe Mardešić and Andrei Prasolov asked whether $\widetilde{H}_\ast$ is additive. (1) and (2) imply $\widetilde{H}_\ast$ is additive on CW, and for finite sums, respectively. So Mardešić and Prasolov began by considering the strong homology of an infinite countable sum $Y$ of Hawaiian earrings $X$. $\widetilde{H}_2(X) = 0$. Mardešić and Prasolov directly computed, though, that $\widetilde{H}_2(X)$ is the quotient of coherent families by trivial families; $\widetilde{H}_2(X) = 0$, in other words, iff every coherent family is trivial. But this, as they and others would show, is a question independent of ZFC.
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