Strong Homology

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Theorem (Eilenberg Steenrod 1945)

Exactly one homology theory H_* satisfies the Eilenberg-Steenrod axioms on the category of finite CW-complexes.

Additivity

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Formal definition [edit]

The Eilenberg–Steenrod axioms apply to a sequence of functors H_n from the category of pairs (X, A) of topological spaces to the category of abelian groups, together with a natural transformation $\partial : H_i(X, A) \to H_{i-1}(A)$ called the **boundary map** (here $H_{i-1}(A)$ is a shorthand for $H_{i-1}(A, \varnothing)$). The axioms are:

1. **Homotopy**: Homotopic maps induce the same map in homology. That is, if $g : (X, A) \to (Y, B)$ is homotopic to $h : (X, A) \to (Y, B)$, then their induced maps are the same.
2. **Excision**: If (X, A) is a pair and U is a subset of X such that the closure of U is contained in the interior of A, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism in homology.
3. **Dimension**: Let P be the one-point space; then $H_n(P) = 0$ for all $n \neq 0$.
4. **Additivity**: If $X = \coprod_\alpha X_\alpha$, the disjoint union of a family of topological spaces X_α, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.
5. **Exactness**: Each pair (X, A) induces a long exact sequence in homology, via the inclusions $i : A \to X$ and $j : X \to (X, A)$: $\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$.

If P is the one point space then $H_0(P)$ is called the **coefficient group**. For example, singular homology (taken with integer coefficients, as is most common) has as coefficients the integers.

Additivity

Definition

A homology theory H_* is *additive* if for every n and every

$$X = \coprod_{\alpha \in A} X_\alpha$$

the inclusion maps $i_\alpha : X_\alpha \to X$ induce an isomorphism

$$i_* : \bigoplus_{\alpha \in A} H_n(X_\alpha) \to H_n(\coprod_{\alpha \in A} X_\alpha)$$

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Henceforth let H_* denote this (unique) extension to CW.

Additivity

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Coherence vs. Triviality