Systems of Filters

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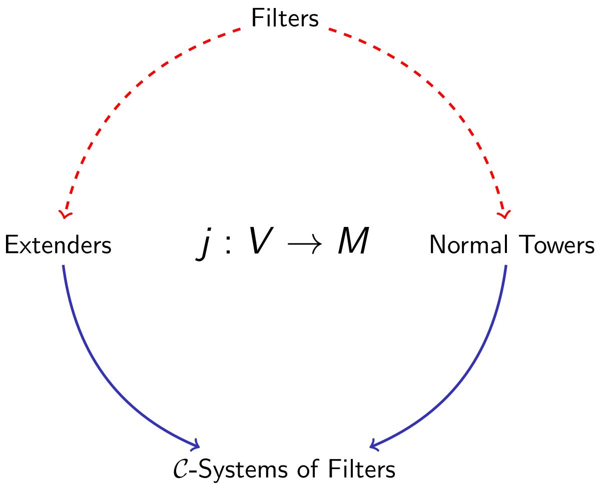
How can we express properties of elementary embeddings?

$F \subseteq \mathcal{P}(X)$ is a filter on X, if F is closed under supersets and finite intersections.



Why Systems of Filters?

Systems of Filters:

- generalize both extenders and normal towers;
- provide a common framework in which properties of extenders and towers can be expressed in a coincise way.

What is a system of filters?

Indices

Extenders

$$\mathbb{E} = \{F_a : a \in [\lambda]^{<\omega}\}$$

Normal Towers

$$\mathbb{T} = \{F_a : a \in V_\lambda\}$$

$\mathcal{C}$-systems of filters

$$\mathbb{S} = \{F_a : a \in \mathcal{C}\}$$

A set $\mathcal{C} \in V$ is a directed set of domains iff the following holds:

1. Ideal property: $\mathcal{C}$ is closed under subsets and unions;
2. Transitivity: $\bigcup \mathcal{C}$ is transitive.

Filter Property and Compatibility

In standard extenders F_a is a filter on $[\kappa_a]^{|a|}$.

$\pi_{ba} : [\kappa_b]^{|b|} \to [\kappa_a]^{|a|}$ is such that given $a, b \in [\lambda]^{<\omega}$ such that $b = \{\alpha_0, \ldots, \alpha_n\} \supseteq a = \{\alpha_{i_0}, \ldots, \alpha_{i_m}\}$ and $s = \{s_0, \ldots, s_n\}$, $\pi_{ba}(s) = \{s_{i_0}, \ldots, s_{i_m}\}$.

For instance if $a = \{1, \omega\}$, $b = \left\{0, 1, 74, \omega, \omega^3 + 1\right\}$, $s = \{0, 1, 2, 3, 4\}$, then $\pi_{ba}(s) = \{1, 3\}$.

We can see F_a as a filter on ${}^a\kappa_a$.

In this case $\pi_{ba} : {}^b\kappa_b \to {}^a\kappa_a$ is just the restriction of functions, i.e. $\pi_{ba}(f) = f \restriction a$.

Filter Property and Compatibility

In standard towers F_a is a filter on $\mathcal{P}(a)$.

$\pi_{ba} : \mathcal{P}(b) \to \mathcal{P}(a)$ is such that given $a, b \in V_\lambda$, $X \in \mathcal{P}(b)$, $\pi_{ba}(X) = X \cap a$.

We can see F_a as a filter on $\{\pi_M : M \subseteq a\}$ (where $\pi_M : M \to V$ is the Mostowski collapse of the structure $(M, \in)$).

In this case π_{ba} is just the restriction of functions, i.e. $\pi_{ba}(f) = f \restriction a$.

Filter Property and Compatibility

Extenders

$\mathbb{E} = \{F_a : a \in [\lambda]^{<\omega}\}$

F_a filter on $[\kappa_a]^{|a|}$

$\pi_{ba}(s) = s_a^b$

$A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$

Normal Towers

$\mathbb{T} = \{F_a : a \in V_\lambda\}$

F_a filter on $\mathcal{P}(a)$

$\pi_{ba}(X) = X \cap a$

$A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$

$\mathcal{C}$-systems of filters

$\mathbb{S} = \{F_a : a \in \mathcal{C}\}$

F_a filter on O_a

$\pi_{ba}(f) = f \restriction a$

$A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$

$$O_a = \{\pi_M \restriction (a \cap M) : \; M \subseteq \mathrm{trcl}(a),\; M \in V\}$$

Filter Property and Compatibility

Extenders

$$\mathbb{E} = \{F_a : a \in [\lambda]^{<\omega}\}$$

$$F_a \text{ filter on } {}^a\kappa_a$$

$$\pi_{ba}(f) = f \restriction a$$

$$A \in F_a \text{ iff } \pi_{ba}^{-1}[A] \in F_b$$

Normal Towers

$$\mathbb{T} = \{F_a : a \in V_\lambda\}$$

$$F_a \text{ filter on } \{\pi_M : M \subseteq a\}$$

$$\pi_{ba}(f) = f \restriction a$$

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$\mathcal{C}$-systems of filters

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Fineness

Extenders

For all $x \in a$,

$\{f \in {}^a\kappa_a : x \in a\} \in F_a$

Normal Towers

For all $x \in a$,

$\{X \in \mathcal{P}(a) : x \in X\} \in F_a$

$\mathcal{C}$-systems of filters

For all $x \in a$,

$\{f \in O_a : x \in \mathrm{dom}(f)\} \in F_a$

Normality

Extenders:

- $u : A \to V$ is regressive on $A \subseteq {}^a\kappa_a$ iff there exists $\alpha \in a$ such that for all $f \in A$, $u(f) \in f(\alpha)$.
- $u : A \to V$ is guessed on $B \subseteq {}^b\kappa_b$, $b \supseteq a$ iff there is a $\beta \in b$ such that for all $f \in B$, $u(\pi_{ba}(f)) = f(\beta)$.

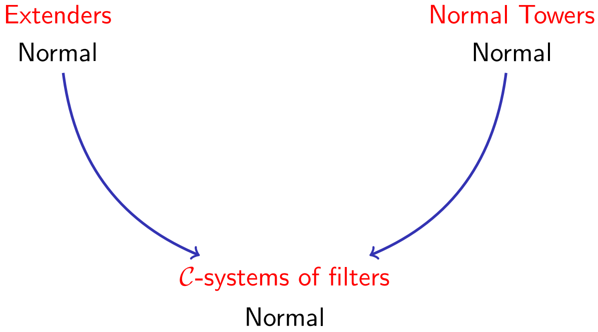
Towers:

- $u : A \to V$ is regressive on $A \subseteq \mathcal{P}(a)$ iff for all $X \in A$, $u(X) \in X$.
- $u : A \to V$ is guessed on $B \subseteq \mathcal{P}(b)$, $b \supseteq a$ iff there is a $y \in b$ such that for all $X \in B$, $u(\pi_{ba}(X)) = y$.

$\mathcal{C}$-System of Filters: define $x \trianglelefteq y$ as $x \in y \vee x = y$.

- $u : A \to V$ is regressive on $A \subseteq O_a$ iff for all $f \in A$, $u(f) \trianglelefteq f(x_f)$ for some $x_f \in \mathrm{dom}(f)$.
- $u : A \to V$ is guessed on $B \subseteq O_b$, $b \supseteq a$ iff there is a $y \in b$ such that for all $f \in B$, $u(\pi_{ba}(f)) = f(y)$.

Normality



(Normality) Every function $u : A \to V$ in V that is regressive on a set $A \in I_a^+$ for some $a \in \mathcal{C}$ is guessed on a set $B \in I_b^+$ for some $b \in \mathcal{C}$ such that $B \subseteq \pi_{ba}^{-1}[A]$;

Ultrafilter Property

Extenders

$\mathbb{E} = \{F_a : a \in [\lambda]^{<\omega}\}$

F_a ultrafilter on ${}^a\kappa_a$

Normal Towers

$\mathbb{T} = \{F_a : a \in V_\lambda\}$

F_a ultrafilter on $\mathcal{P}(a)$

$\mathcal{C}$-systems of filters

$\mathbb{S} = \{F_a : a \in \mathcal{C}\}$

F_a ultrafilter on O_a

$$O_a = \{\pi_M \restriction (a \cap M) : \ M \subseteq \mathrm{trcl}(a),\ M \in V\}$$

$\langle\kappa,\lambda\rangle$-Systems of Filters

$\langle\kappa,\lambda\rangle$-Extenders: F_a is κ-complete. The support κ_a is the least ξ such that $[\xi]^{|a|} \in F_a$. And

- if $a \subseteq b \in [\lambda]^{<\omega}$ then
  - $\kappa_a \leq \kappa_b$;
  - if $\max(a) = \max(b)$, then $\kappa_a = \kappa_b$;
- $\kappa_{\{\kappa\}} = \kappa$;

$\mathbb{S}$ is a $\langle\kappa,\lambda\rangle$-system of filters if:

κ_a is the support of a iff it is the minimum ξ such that $O_a \cap {}^a V_\xi \in F_a$.

- $\mathrm{rank}(\mathcal{C}) = \lambda$ and $\kappa \subseteq \bigcup \mathcal{C}$,
- $F_{\{\gamma\}}$ is principal generated by $\mathrm{id} \restriction \{\gamma\}$ whenever $\gamma < \kappa$,
- $\kappa_a \leq \kappa$ whenever $a \in V_{\kappa+2}$.

From a system of ultrafilters to elementary embeddings

Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, and define

$$U_{\mathcal{S}} = \{u : O_a \to V \ : \ a \in \mathcal{C}\}$$

and the relations

$$u =_{\mathcal{S}} v \Leftrightarrow \{f \in O_c : \ u(\pi_{ca}(f)) = v(\pi_{cb}(f))\} \in F_c$$

$$u \in_{\mathcal{S}} v \Leftrightarrow \{f \in O_c : \ u(\pi_{ca}(f)) \in v(\pi_{cb}(f))\} \in F_c$$

where $O_a = \mathrm{dom}(u)$, $O_b = \mathrm{dom}(v)$, $c = a \cup b$.

The ultrapower of V by $\mathcal{S}$ is $\mathrm{Ult}(V, \mathcal{S}) = \langle U_{\mathcal{S}} / =_{\mathcal{S}}, \ \in_{\mathcal{S}} \rangle$.

Define $j_{\mathcal{S}} : V \to \mathrm{Ult}(V, \mathcal{S})$ by $j_{\mathcal{S}}(x) = [c_x]_{\mathcal{S}}$, $c_x : O_\emptyset \to \{x\}$.

From elementary embedding to system of ultrafilters

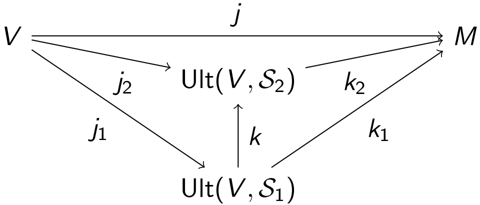
Let $j : V \to M \subseteq V[G]$ be a generic elementary embedding, $\mathcal{C} \in V$ be a directed set of domains such that $(j \restriction a)^{-1} \in M$ for all $a \in \mathcal{C}$.

The $\mathcal{C}$-system of ultrafilters derived from j is $\mathcal{S} = \langle F_a : \ a \in \mathcal{C} \rangle$ such that:

$$F_a = \left\{ A \subseteq O_a : (j \restriction a)^{-1} \in j(A) \right\}.$$

$\mathcal{C}$-systems of filters from a single j

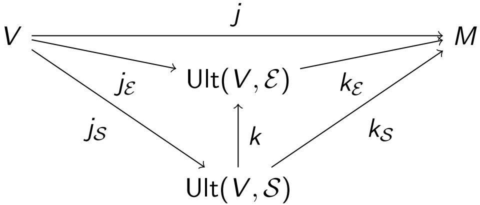
Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in W, $\mathcal{S}_n$ be the $\mathcal{C}_n$-system of V-ultrafilters derived from j for $n = 1, 2$, $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Then $\mathrm{Ult}(V, \mathcal{S}_2)$ can be factored into $\mathrm{Ult}(V, \mathcal{S}_1)$, and $\mathrm{crit}(k_1) \leq \mathrm{crit}(k_2)$ where k_1, k_2 are the corresponding factor maps.



$\mathcal{C}$-systems of filters from a single j

non(F)$=\min\{|A| : A \in F\}$. non($\mathcal{S}$)$=\sup\{\text{non}(F_a)+1 : a \in \mathcal{C}\}$.

Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in W, $\mathcal{S}$ be the $\mathcal{C}$-system of filters derived from j, $\mathcal{E}$ be the extender of length $\lambda \supseteq j[\text{non}(\mathcal{S})]$ derived from j. Then $\text{Ult}(V, \mathcal{E})$ can be factored into $\text{Ult}(V, \mathcal{S})$, and $\text{crit}(k_{\mathcal{S}}) \leq \text{crit}(k_{\mathcal{E}})$.



Generic $\mathcal{C}$-Systems of ultrafilters

- Let $\dot{F}$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Define

$$\mathbf{I}(\dot{F}) = \left\{ Y \subset X : \left[\!\left[\check{Y} \in \dot{F} \right]\!\right] = 0_{\mathbb{B}} \right\}$$

- Let I be an ideal in V on $\mathcal{P}(X)$ and consider the poset $\mathbb{B} = \mathcal{P}(X)/I$. Let $\dot{\mathbf{F}}(I)$ be the $\mathbb{B}$-name defined by

$$\dot{\mathbf{F}}(I) = \{\langle \check{Y}, [Y]_I \rangle : \ Y \subseteq X\}$$

- Let $\dot{\mathcal{S}}$ be a $\mathbb{B}$-name for a $\mathcal{C}$-system of ultrafilters. Then we define the corresponding $\mathcal{C}$-system of filters in V, $\mathbf{I}(\dot{\mathcal{S}})$.
- Conversely, $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. Then we define the corresponding name for a $\mathcal{C}$-system of ultrafilters, $\dot{\mathbf{F}}(\mathbb{S})$.

Generic $\mathcal{C}$-Systems of ultrafilters

Let $\mathbb{S}$ be a $\langle\kappa,\lambda\rangle$-$\mathcal{C}$-system of filters, $\mathbb{C}$ be a κ-cc cBa. Define $\mathbb{S}^{\mathbb{C}} = \left\{F_a^{\mathbb{C}} :\ a \in \mathcal{C}\right\}$ where $F_a^{\mathbb{C}} = \left\{A \subseteq (O_a)^{V^{\mathbb{C}}} :\ \exists B \in \check{F}_a\ A \supseteq B\right\}$.

$\mathbb{S}^{\mathbb{C}}$ is a $\mathcal{C}$-system of filters, $\mathbb{C} * \mathbb{S}^{\mathbb{C}}$ is isomorphic to $\mathbb{S} * j(\mathbb{C})$ and the following diagram commutes.

$$\begin{array}{ccccccc}
V & \xrightarrow{j_{\dot{\mathbf{F}}(\mathbb{S})}} & M & \subseteq & V^{\mathbb{S}} & & \\
\cap\!\! | & & \cap\!\! | & & \cap\!\! | & & \\
V^{\mathbb{C}} & \xrightarrow[j_{\dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}})}]{} & M^{j(\mathbb{C})} & \subseteq & V^{\mathbb{S}*j(\mathbb{C})} & = & V^{\mathbb{C}*\mathbb{S}^{\mathbb{C}}}
\end{array}$$

Generic $\mathcal{C}$-Systems of ultrafilters

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Thank you!