REVERSIBILITY OF RELATIONAL STRUCTURES

Miloš Kurilić and Nenad Morača

Department of Mathematics and Informatics, Faculty of Sciences,
University of Novi Sad, Serbia

2nd February 2016

Plan of the presentation

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- Preliminaries

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- Strongly reversible, reversible and weakly reversible interpretations

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- Strongly reversible, reversible and weakly reversible interpretations
- Characterization of strongly reversible interpretations

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- From now on, we work with the **binary language** $L_b = \langle R \rangle, \mathrm{ar}(R) = 2$. A binary structure $\langle X, \rho \rangle$ is **connected** iff $|X/\rho_{rst}| = 1$, where ρ_{rst} is the smallest equivalence relation containing ρ.
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	$\neg$ (b1) ($\top$)	(b1)	(b2)	(b3)	(b4)
$\neg$ (a1) ($\top$)	$\bigcup_\omega \mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_2^0 \cup \bigcup_\omega \mathbb{L}_4$	$\bigcup_\omega \mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_4 \cup \mathbb{L}_4^0$	$\mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_2^0 \cup \bigcup_\omega \mathbb{L}_4^{0,2}$	$\bigcup_\omega \mathbb{L}_2^0 \cup \mathbb{L}_2^1 \cup \bigcup_\omega \mathbb{L}_4^{0,2}$	$\bigcup_\omega \mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_4$
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(a1)	$\mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_\omega \cup \bigcup_\omega \mathbb{L}_\omega^0$	$\bigcup_\omega \mathbb{L}_2^0 \cup \bigcup_\omega \mathbb{L}_4^0 \cup \mathbb{L}_4^{0,2}$	$\mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_2^0 \cup \bigcup_\omega \mathbb{L}_4^0$	$\bigcup_\omega \mathbb{L}_2^0 \cup \mathbb{L}_2^1 \cup \bigcup_\omega \mathbb{L}_4^{0,3}$	$\mathbb{L}_2 \cup \bigcup_\omega \mathbb{L}_\omega$
(a2)	$\bigcup_\omega \mathbb{L}_\omega \cup \bigcup_\omega \mathbb{L}_\omega^0$	$\bigcup_\omega \mathbb{L}_\omega \cup \mathbb{L}_\omega^0$	$\mathbb{L}_\omega \cup \bigcup_\omega \mathbb{L}_\omega^0$	$\bigcup_\omega \mathbb{L}_\omega^0 \cup \mathbb{L}_\omega^1$	$\bigcup_\omega \mathbb{L}_\omega$
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For $k\in\omega$ and $*\in\{\prime,\#,\pm,=,\rightleftarrows,\Delta\}$, let $\mathbb{Y}_i^{\theta^*=k},\ i\in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^*=k$, and let $\mathbb{Z}_i^{\theta^*=k},\ i\in N_k^*$, be all nonisomorphic...

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$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2} \cdot \lvert(\rho \setminus \Delta_X) \cap (\rho \setminus \Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

- $* \in \{\prime\}$ ... connected structures for which $\theta' = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2}\cdot\lvert(\rho\backslash\Delta_X) \cap (\rho\backslash\Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

- $* \in \{\prime\}$ ... connected structures for which $\theta' = k$.
- $* \in \{\#\}$ ... digraph trees for which $\theta^{\#} = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2} \cdot \lvert(\rho \setminus \Delta_X) \cap (\rho \setminus \Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

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- $* \in \{\pm\}$ ... connected ($\deg^{\pm}$-)regular digraphs for which $\theta^{\pm} = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho\cap\Delta_X\rvert$	$\frac{1}{2}\cdot\lvert(\rho\setminus\Delta_X)\cap(\rho\setminus\Delta_X)^{-1}\rvert$	$\cdots$

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- $* \in \{\prime\}$ ... connected structures for which $\theta' = k$.
- $* \in \{\#\}$ ... digraph trees for which $\theta^{\#} = k$.
- $* \in \{\pm\}$ ... connected ($\deg^{\pm}$-)regular digraphs for which $\theta^{\pm} = k$.
- $* \in \{=\}$ ... reflexive digraph trees for which $\theta^{=} = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2}\cdot\lvert(\rho\setminus\Delta_X) \cap (\rho\setminus\Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

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- $* \in \{\pm\}$ ... connected ($\mathrm{deg}^{\pm}$-)regular digraphs for which $\theta^{\pm} = k$.
- $* \in \{=\}$ ... reflexive digraph trees for which $\theta^{=} = k$.
- $* \in \{\rightleftarrows\}$ ... connected graphs for which $\theta^{\rightleftarrows} = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2} \cdot \lvert(\rho \setminus \Delta_X) \cap (\rho \setminus \Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

- $* \in \{\prime\}$ ... connected structures for which $\theta' = k$.
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- $* \in \{\pm\}$ ... connected ($\mathrm{deg}^{\pm}$-)regular digraphs for which $\theta^{\pm} = k$.
- $* \in \{=\}$ ... reflexive digraph trees for which $\theta^{=} = k$.
- $* \in \{\rightleftarrows\}$ ... connected graphs for which $\theta^{\rightleftarrows} = k$.
- $* \in \{\Delta\}$ ... poset trees with no leaves on first level, for which $\theta^{\Delta} = k$.

Further examples of reversible disconnected structures

$*$	$\#$	$\prime$	$\pm$	$=$	$\rightleftarrows$	$\Delta, \circ$
$\theta^*(\mathbb{X})$	$\lvert X\rvert$	$\lvert\rho\rvert$	$\mathrm{Deg}^{\pm}(\mathbb{X})$	$\lvert\rho \cap \Delta_X\rvert$	$\frac{1}{2} \cdot \lvert(\rho \setminus \Delta_X) \cap (\rho \setminus \Delta_X)^{-1}\rvert$	$\cdots$

For $k \in \omega$ and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let $\mathbb{Y}_i^{\theta^*=k}$, $i \in M_k^*$, be all finite and finitely many infinite (if there are any) nonisomorphic connected reversible structures for which $\theta^* = k$, and let $\mathbb{Z}_i^{\theta^*=k}$, $i \in N_k^*$, be all nonisomorphic...

- $* \in \{\prime\}$ ... connected structures for which $\theta' = k$.
- $* \in \{\#\}$ ... digraph trees for which $\theta^{\#} = k$.
- $* \in \{\pm\}$ ... connected ($\deg^{\pm}$-)regular digraphs for which $\theta^{\pm} = k$.
- $* \in \{=\}$ ... reflexive digraph trees for which $\theta^{=} = k$.
- $* \in \{\rightleftarrows\}$ ... connected graphs for which $\theta^{\rightleftarrows} = k$.
- $* \in \{\Delta\}$ ... poset trees with no leaves on first level, for which $\theta^{\Delta} = k$.

Then all the components $\mathbb{Z}_i^{\theta^*=k}$ are connected, finite and reversible, except the components $\mathbb{Z}_i^{\theta^{\pm}=k}$ which are at most countable and reversible.

Further examples of reversible disconnected structures

Further examples of reversible disconnected structures

$*$	k	$\lvert Y_i^{\theta^*=k}\rvert$	$\lvert M_k^*\rvert$	$\lvert Z_i^{\theta^*=k}\rvert$	$\lvert N_k^*\rvert$
$\prime$	$k \in \omega$	$< \omega$	$< \omega$	$< \omega$	$< \omega$
$\#$	$k \in \mathbb{N}$	$< \omega$	$< \omega$	$< \omega$	$< \omega$
$\pm$	$k \leq 1$	≤ 2	≤ 2	≤ 2	≤ 2
	$k \geq 2$	$\leq \omega$	ω	$\leq \omega$	$\mathfrak{c}$
$=$	$k \in \omega$	$< \infty$	ω	$< \omega$	$< \omega$
$\leftrightarrows$	$k \in \omega$	$< \infty$	ω	$< \omega$	$< \omega$
Δ	$k \in \omega$	$< \infty$	ω	$< \omega$	$< \omega$

Table 2: The size of the sets $Y_i^{\theta^*=k}$, M_k^*, $Z_i^{\theta^*=k}$ and N_k^*

Further examples of reversible disconnected structures

Further examples of reversible disconnected structures

For $n \geq 2$, and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let us define the following structure:

Further examples of reversible disconnected structures

For $n \geq 2$, and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let us define the following structure:

$$\mathbb{X}_n^* := \bigcup_{k=0}^{n-1} \Big(\bigcup_{i \in M_k^*} \mathbb{Y}_i^{\theta^*=k} \Big) \cup \bigcup_{i \in N_n^*} \Big(\bigcup_{\omega} \mathbb{Z}_i^{\theta^*=n} \Big).$$

Further examples of reversible disconnected structures

For $n \geq 2$, and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let us define the following structure:

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All the structures $\mathbb{X}_n^*$ are reversible by the last corollary.

Further examples of reversible disconnected structures

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All the structures $\mathbb{X}_n^*$ are reversible by the last corollary.

	$\mathbb{X}_n^{\prime}$	$\mathbb{X}_n^{\#}$	$\mathbb{X}_n^{\pm}$	$\mathbb{X}_n^{=}$	$\mathbb{X}_n^{\leftrightarrows}$	$\mathbb{X}_n^{\Delta}$
$n = 2$	(a4) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)
$n \geq 3$	(a1) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)

Table 3: The place of the structures $\mathbb{X}_n^*$ in Table 1

Further examples of reversible disconnected structures

For $n \geq 2$, and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let us define the following structure:

$$\mathbb{X}_n^* := \bigcup_{k=0}^{n-1} \Big(\bigcup_{i \in M_k^*} \mathbb{Y}_i^{\theta^* = k} \Big) \cup \bigcup_{i \in N_n^*} \Big(\bigcup_{\omega} \mathbb{Z}_i^{\theta^* = n} \Big).$$

All the structures $\mathbb{X}_n^*$ are reversible by the last corollary.

	$\mathbb{X}_n^\prime$	$\mathbb{X}_n^\#$	$\mathbb{X}_n^\pm$	$\mathbb{X}_n^=$	$\mathbb{X}_n^\leftrightarrows$	$\mathbb{X}_n^\Delta$
$n = 2$	(a4) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)
$n \geq 3$	(a1) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)

Table 3: The place of the structures $\mathbb{X}_n^*$ in Table 1

Most of the structures $\mathbb{X}_n^*$ are placed in (a1) row of Table 1. The condition (a1) is not operative, therefore the presented sufficient conditions do not substitute each other.

Further examples of reversible disconnected structures

For $n \geq 2$, and $* \in \{\prime, \#, \pm, =, \rightleftarrows, \Delta\}$, let us define the following structure:

$$\mathbb{X}_n^* := \bigcup_{k=0}^{n-1} \Big(\bigcup_{i \in M_k^*} \mathbb{Y}_i^{\theta^*=k} \Big) \cup \bigcup_{i \in N_n^*} \Big(\bigcup_{\omega} \mathbb{Z}_i^{\theta^*=n} \Big).$$

All the structures $\mathbb{X}_n^*$ are reversible by the last corollary.

	$\mathbb{X}_n^{\prime}$	$\mathbb{X}_n^{\#}$	$\mathbb{X}_n^{\pm}$	$\mathbb{X}_n^{=}$	$\mathbb{X}_n^{\leftrightarrows}$	$\mathbb{X}_n^{\Delta}$
$n = 2$	(a4) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)
$n \geq 3$	(a1) (b2)	(a4) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)	(a1) (b2)

Table 3: The place of the structures $\mathbb{X}_n^*$ in Table 1

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References

- P. H. Doyle, J. G. Hocking, Bijectively related spaces, I. Manifolds. Pac. J. Math. 111 (1984) 23–33.
- R. Fraïssé, Theory of relations, Revised edition, With an appendix by Norbert Sauer, Studies in Logic and the Foundations of Mathematics, 145. North-Holland, Amsterdam, (2000)
- C. W. Henson, A family of countable homogeneous graphs, Pacific J. Math., 38,1 (1971) 69–83.
- W. Hodges, Model theory, Encyclopedia of Mathematics and its Applications, 42, Cambridge University Press, Cambridge, 1993.
- M. Kukieła, Reversible and bijectively related posets, Order 26 (2009) 119–124.
- M. S. Kurilić, Reversibility of topological spaces, (unpublished manuscript)
- A. H. Lachlan, R. E. Woodrow, Countable ultrahomogeneous undirected graphs, Trans. Amer. Math. Soc., 262,1 (1980) 51–94.
- R. C. Lyndon, Properties preserved under homomorphism, Pacific J. Math. 9 (1959) 143–154.
- M. Rajagopalan, A. Wilansky, Reversible topological spaces, J. Aust. Math. Soc. 61 (1966) 129–138.
- J. G. Rosenstein, Linear orderings, Pure and Applied Mathematics, 98, Academic Press, Inc., Harcourt Brace Jovanovich Publishers, New York-London, 1982.
- J. H. Schmerl, Countable homogeneous partially ordered sets, Algebra Univers. 9,3 (1979) 317-321.
- P. Vopěnka, A. Pultr, Z. Hedrlín, A rigid relation exists on any set, Comment. Math. Univ. Carolinae 6 (1965) 149–155.

Thank you for your attention.