# The Bolzano property

Przemysław Tkacz

Cardnial Stefan Wyszyski University in Warsaw

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# Introduction

**Theorem (Bolzano 1817)**

*If a continuous $f : [a, b] \to R$ and*

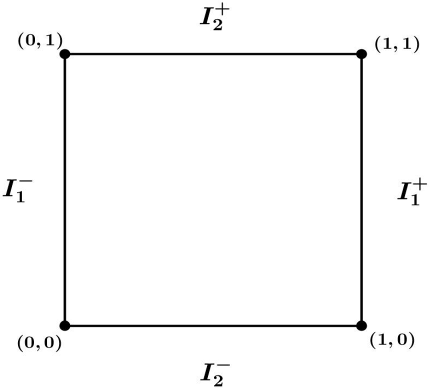
$$f(a) \cdot f(b) \leq 0,$$

*then there is $c \in [a, b]$ such that $f(c) = 0$.*

# Introduction

$I^n = [0,1]^n$ : $n$-dimensional cube in $\mathbf{R}^n$.
Its *i-th opposite faces* are defined as follows:

$$I_i^- := \{x \in I^n : \ x(i) = 0\}, \ I_i^+ := \{x \in I^n : \ x(i) = 1\}$$



# The Poincaré-Miranda theorem

**Theorem (Poincaré 1883)**

*If a continuous*

$$f = (f_1, f_2, \dots, f_n) : I^n \to \mathbb{R}^n,$$

$$f_i(I_i^-) \subset (-\infty, 0], \qquad f_i(I_i^+) \subset [0, \infty),$$

*then there is* $c \in I^n$ *such that* $f(c) = (0, 0, \dots, 0)$

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**Theorem (Miranda 1940)**

*The Poincaré theorem is equivalent to the Brouwer fixed point theorem.*

# The $n$-dimensional Bolzano property

**Definition (Kulpa 1994)**

The topological space $X$ has *the n-dimensional Bolzano property* if there exists a family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of non-empty disjoint closed subsets such that for every continuous

$$f = (f_1, \ldots, f_n) : X \to R^n,$$

$$\forall_{i \leq n} \; f_i(A_i) \subset (-\infty, 0], \text{ and } f_i(B_i) \subset [0, \infty),$$

there exists $c \in X$ such that $f(c) = 0$.

$\{(A_i, B_i) : i = 1, \ldots, n\}$ : *an n-dimensional boundary system*

# The $n$-dimensional Bolzano property

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$$\text{for each } i \leq n \; A_i \subset H_i^-, B_i \subset H_i^+ \text{ and } H_i^- \cup H_i^+ = X$$

we have

$$\bigcap \{H_i^- \cap H_i^+ : i = 1, \ldots, n\} \neq \emptyset.$$

# The $n$-dimensional Bolzano property

**Theorem**

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**Theorem**

*If $X$ is perfectly normal and has the Kulpa n-dimensional Bolzano property. Than $X$ has the n-dimensional Bolzano property.*

# Properties

**Theorem**

*Let* $\{(A_i, B_i) : i = 1, ..., n\}$ *be the n-dimensional boundary system in* $T_5$ *space* $X$. *Then for each* $i \leq n$ $A_i, B_i$ *have an* $(n-1)$*-dimensional Bolzano property.*
*Moreover the families*

$$\{(A_i \cap A_j, A_i \cap B_j) : j \neq i\}, \{(B_i \cap A_j, B_i \cap B_j) : j \neq i\}$$

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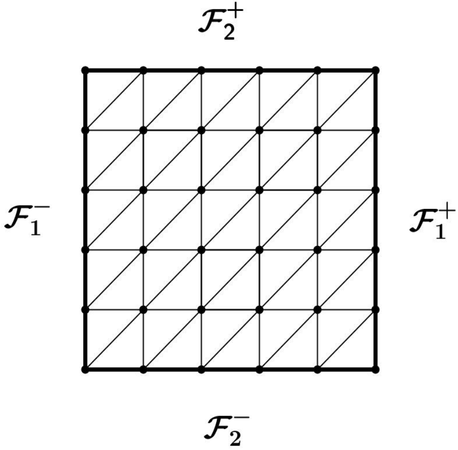
## Corollary

*Let $I_1, I_2 \subset \{1, \ldots, n\}$, $I_1 \cap I_2 = \emptyset$. Then the subspace*

$$\bigcap_{i \in I_1} A_i \cap \bigcap_{i \in I_2} B_i$$

*has an $(n - (card(I_1) + card(I_2)))$-dimensional Bolzano property.*

# An $n$-cube-like polyhedron



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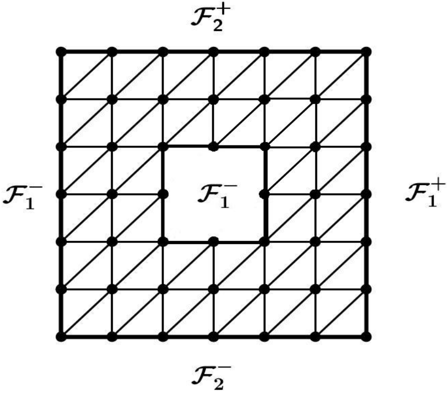

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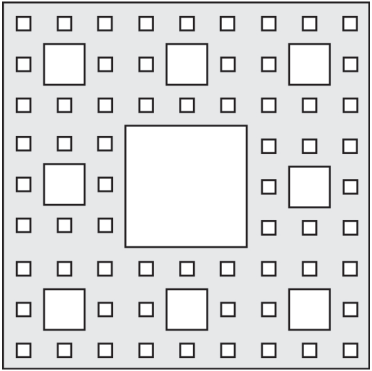

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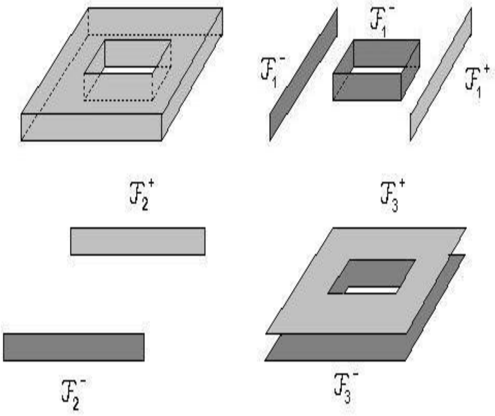

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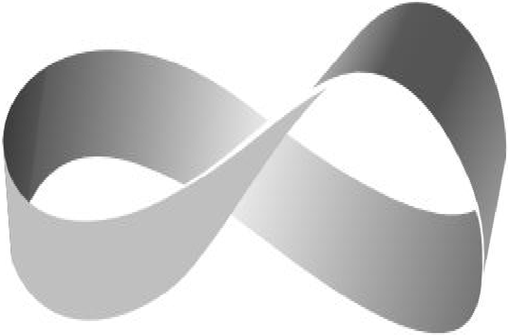

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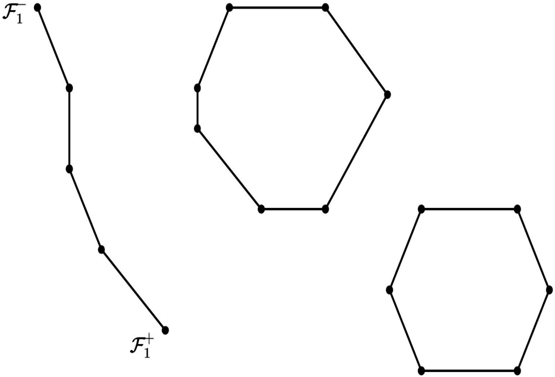

# An $n$-cube-like polyhedron

**Theorem**

*Let $(\bar{K}, \bar{\mathcal{K}})$ be an n-cube-like polyhedron in $R^m$. Then $\bar{K}$ has an n-dimensional Bolzano property.*

# The Steinhaus chains

**Theorem (PT and Turzański 2008)**

*For an arbitrary decomposition of n-dimensional cube $I^n$ onto $k^n$ cubes and an arbitrary coloring function $F\colon T(k) \to \{1, ...n\}$ for some natural number $i \in \{1, ...n\}$ there exists an i-th colored chain $P_1, ..., P_r$ such that*

$$P_1 \cap I_i^+ \neq \emptyset \text{ and } P_r \cap I_i^- \neq \emptyset.$$

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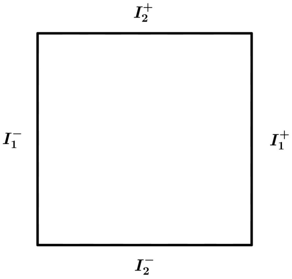
"1"-white "2"-black



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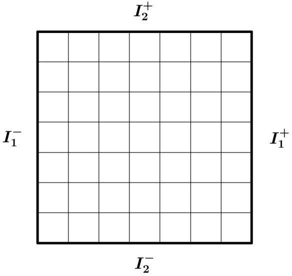

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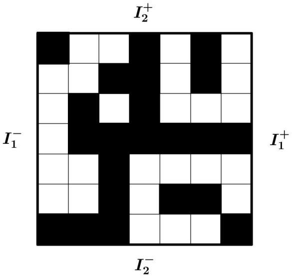

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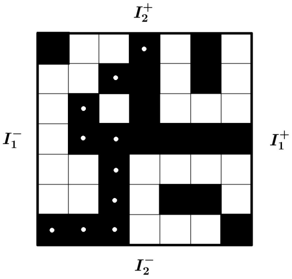

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## Theorem (Topological version)

*Let $\{U_i : i = 1, \cdots, n\}$ be an open covering of $I^n$. Then for some $i \in \{1, ...n\}$ there exists continuum $W \subset U_i$ such that*

$$W \cap I_i^- \neq \emptyset \neq W \cap I_i^+.$$

# The Steinhaus chains

**Theorem (PT and Turzański)**

*The following statements are equivalent:*

1. *Theorem(on the existence of a chain)*
2. *The Poincaré theorem*
3. *The Brouwer Fixed Point theorem.*

# The Steinhaus chains

**Theorem (Michalik, P T, Turzański 2015)**

*Let $\mathcal{K}^n$ be an n-cube-like complex. Then for every map $\phi\colon |\mathcal{K}^n| \to \{1, \ldots, n\}$ there exist $i \in \{1, \ldots, n\}$ and i-th colored chain $\{s_1, \ldots, s_m\} \subset |\mathcal{K}^n|$ such that*

$$s_1 \in \mathcal{F}_i^- \text{ and } s_m \in \mathcal{F}_i^+$$

.

(The sequence $\{s_1, \ldots, s_m\} \subset |\mathcal{K}^n|$ is a *chain* if for each $i \in \{1, \ldots, m-1\}$ we have $\{s_i, s_{i+1}\} \in \mathcal{K}^n$.)

# The inverse system

Let us consider the inverse system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ where:
(i) $\forall \sigma \in \Sigma$ $X_\sigma$ is a compact space with $n$-dimensional boundary system $\{(A^\sigma_i, B^\sigma_i) : i = 1, ..., n\}$.
(ii) $\forall \sigma, \rho \in \Sigma, \rho \leq \sigma$ the map $\pi^\sigma_\rho : X_\sigma \to X_\rho$ is a surjection such that $\pi^\sigma_\rho(A^\sigma_i) = A^\rho_i,\ \pi^\sigma_\rho(B^\sigma_i) = B^\rho_i$.

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**Theorem**

*The space $X = \varprojlim\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ has n-dimensional Bolzano property.*

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**Corollary**

*Pseudoarc has the Bolzano property.*

# The Bolzano property and the dimension

**Theorem**

*Let $X$ be a normal space. $dim X \geq n$ iff there exist a family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of non-empty disjoint closed subsets such that for every family $\{L_i : i = 1, \ldots, n\}$ where $L_i$ is a partition between $A_i$ and $B_i$ we have*

$$\bigcap_{i=1}^{n} L_i \neq \emptyset.$$

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**Theorem**

*If a normal space $X$ has n-dimensional Bolzano property. Then $dimX \geq n$.*

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## Theorem

*If a normal space $X$ has n-dimensional Bolzano property. Then $dimX \geq n$.*

## Theorem

*If $X$ is a perfectly normal space $X$ and $dimX \geq n$. Then $X$ has n-dimensional Bolzano property.*

**Problem**

Is there a gap beetween the Bolzano property and the dimension of $X$?