The cube-like complexes and the Poincaré - Miranda theorem

Maria Kidawa and Przemysław Tkacz

Cardinal Stefan Wyszyński University in Warsaw

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The Poincaré-Miranda theorem

Theorem (Poincaré 1883)

If

$$f = (f_1, f_2, \ldots, f_n) : I^n \to \mathbb{R}^n,$$

$$f_i(I_i^-) \subset (-\infty, 0], \qquad f_i(I_i^+) \subset [0, \infty),$$

$$I_i^- := \{x \in I^n : x(i) = -1\}, \qquad I_i^+ := \{x \in I^n : x(i) = 1\},$$

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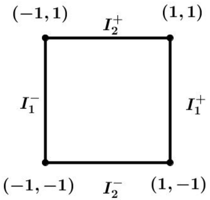

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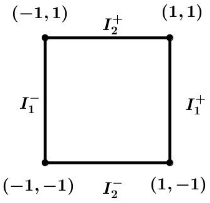

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Problem

Can we generalize the Poincaré theorem?

Definitions

A : finite, nonempty set
$\mathcal{P}_{n+1}(A)$: all subsets of A with cardinality $n+1$
$\mathcal{P}_{n+1}(A) \ni S$: n - *simplex* defined on A
$T \in \mathcal{P}_{k+1}(S)$: k - *face* of an n-simplex S, $k < n$

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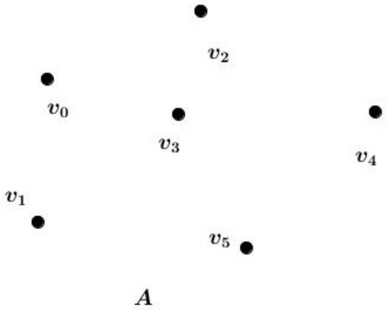

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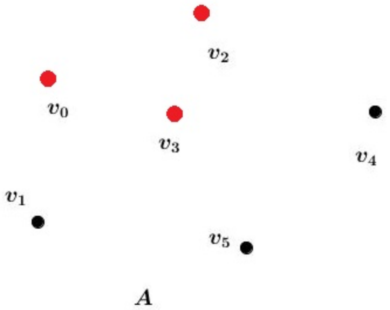

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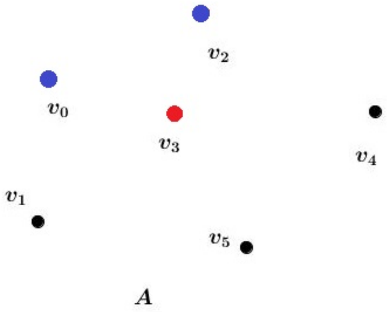

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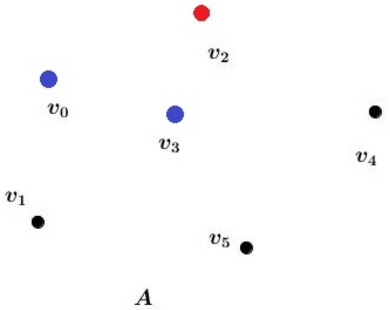

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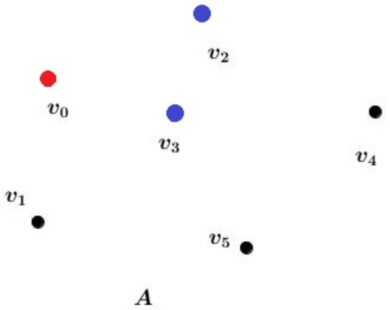

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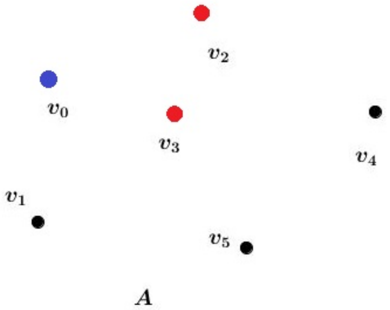

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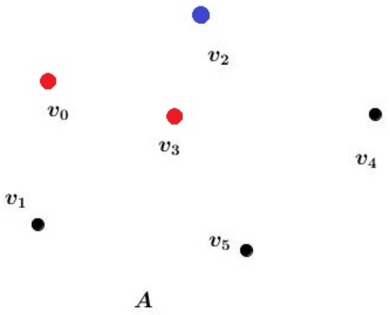

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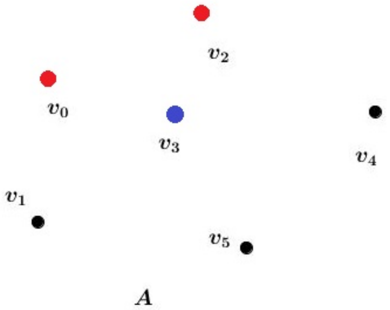

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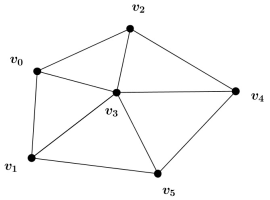

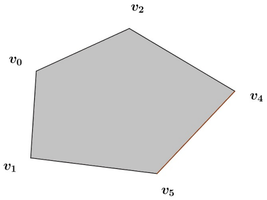

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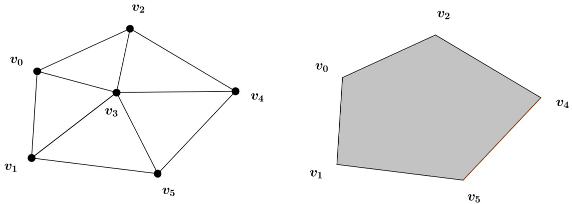

Observation

Each polyhedron determines an abstract complex called its vertex-scheme.

Definitions

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$$\emptyset \neq \mathcal{S} \subset \mathcal{P}(A)$$

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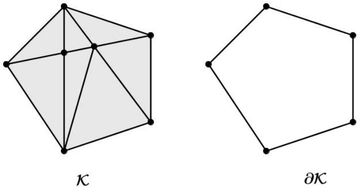
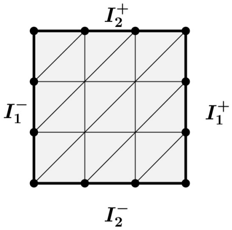
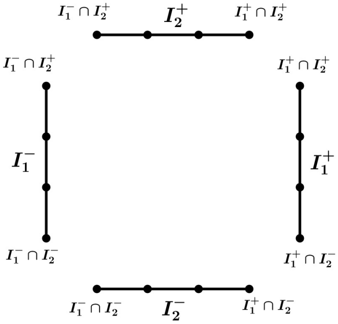


Intuition



I_2^+
I_1^-
I_1^+
I_2^-



$I_1^- \cap I_2^+$
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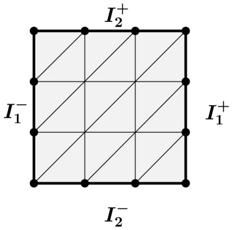
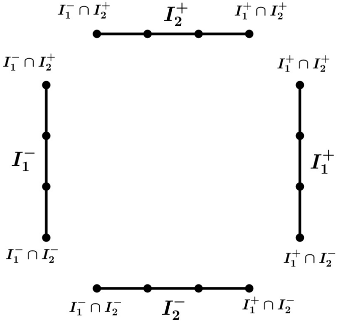




1. $\partial I^2 = \bigcup_{i=1}^{2} I_i^- \cup I_i^+$,
2. Each one of $I_1^-, I_1^+, I_2^-, I_2^+$ is an 1-dimensional cube
3. Opposite faces of an 1- dimensional cube I_i^ε have the folowing form: $I_i^\varepsilon \cap I_j^-$, $I_i^\varepsilon \cap I_j^+$ for $j \neq i$.

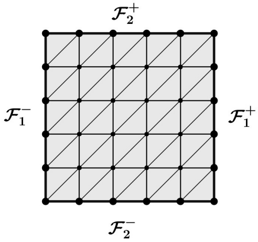
The n-cube like complex

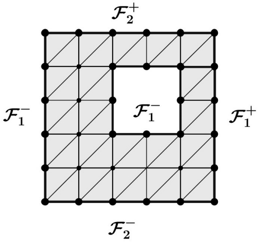
Let $\mathcal{K}^0 = \{a\}$, where $a \in A$.
The complex $\mathcal{K}^n$ generated by $\mathcal{S} \subset \mathcal{P}_{n+1}(A)$ is *an n-cube-like complex*, if:

(A) For all $(n-1)$-face $T \in \mathcal{K}^n \setminus \partial\mathcal{K}^n$ there exist exactly two n-simplexes $S, S' \in \mathcal{K}^n$ such that $S \cap S' = T$.

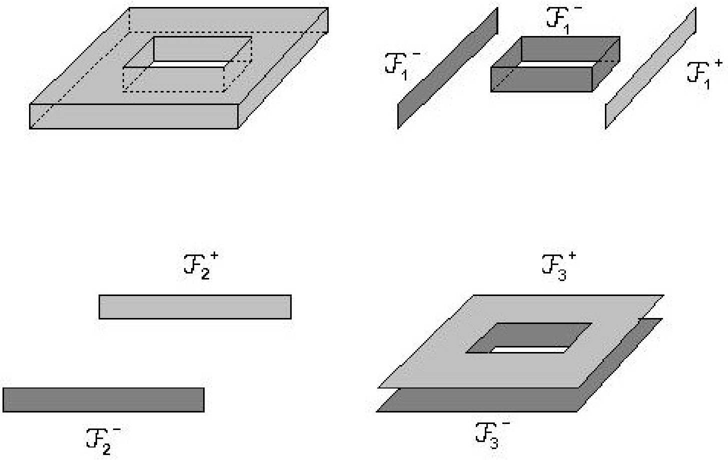
(B) There exist subcomplexes $\mathcal{F}_i^-, \mathcal{F}_i^+$ for $i \in \{1, 2, \dots, n\}$, called *i-th opposite faces* such that:

- (B_1) $\partial\mathcal{K}^n = \bigcup_{i=1}^{n} \mathcal{F}_i^- \cup \mathcal{F}_i^+$
- (B_2) $\mathcal{F}_i^- \cap \mathcal{F}_i^+ = \emptyset$ for $i = \{1, 2, \dots, n\}$
- (B_3) $\forall_{i\in\{1,\dots,n\}}, \forall_{\varepsilon\in\{+,-\}}$ $\mathcal{F}_i^\varepsilon$ is an $(n-1)$-cube-like complex, such that its opposite faces have the following form $\mathcal{F}_i^\varepsilon \cap \mathcal{F}_j^-, \mathcal{F}_i^\varepsilon \cap \mathcal{F}_j^+$, $j \neq i$.





Example


$\mathcal{F}_1^-$
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The construction of an n-cube-like complex

Definition

$S = \{v_0, v_1, \ldots, v_n\}$: an n-simplex; $a, b \in L$.

An *S-doubled complex* $dc(S)_a^b$ is an abstract complex $\mathcal{K}(\mathcal{F}) \subset \mathcal{P}(S \times \{a, b\})$ generated by

$$\mathcal{F} = \big\{\{(v_0, a), \ldots, (v_i, a), (v_i, b), \ldots, (v_n, b)\} \colon i \in \{0, 1, \ldots, n\}\big\}$$

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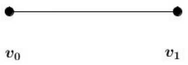
An S-doubled complex in 1-dimensional case.



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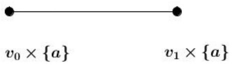

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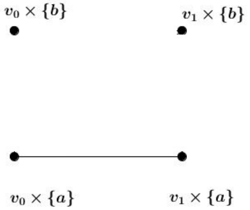

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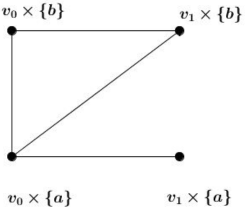

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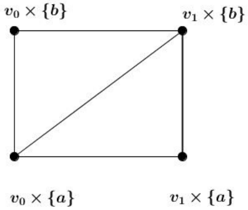

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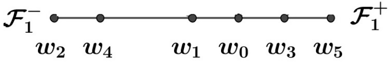

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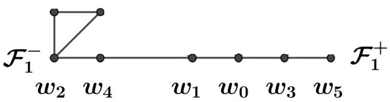

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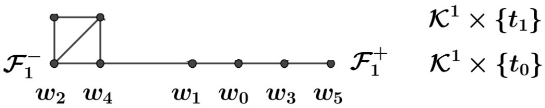

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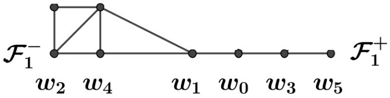

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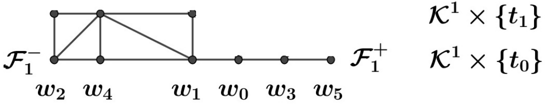

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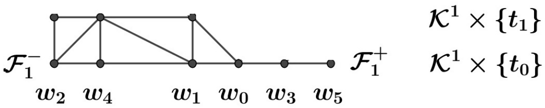

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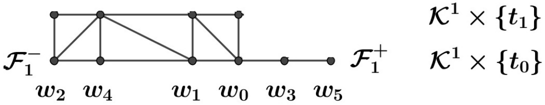

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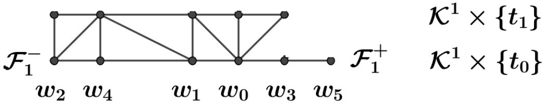

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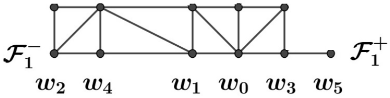

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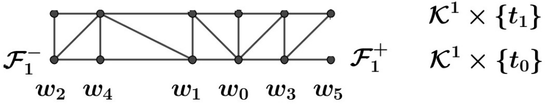

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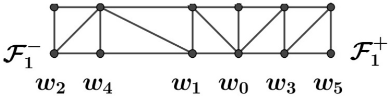

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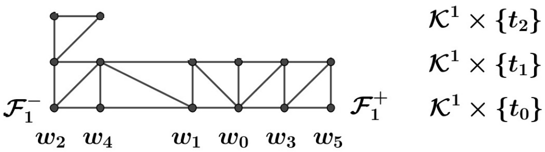

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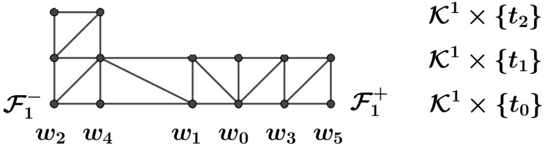

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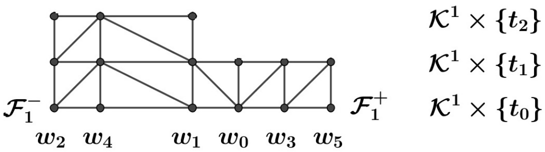

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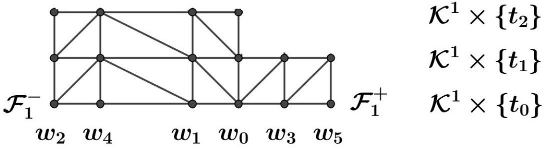

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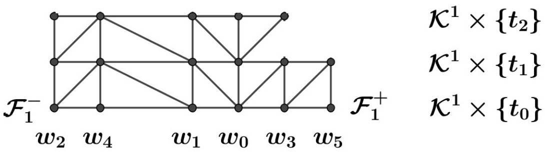

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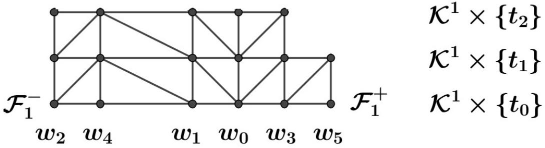

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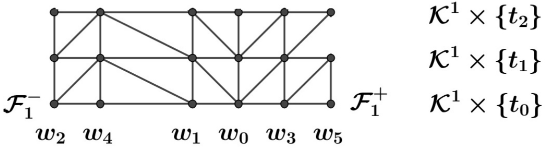

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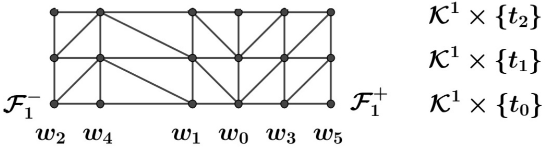

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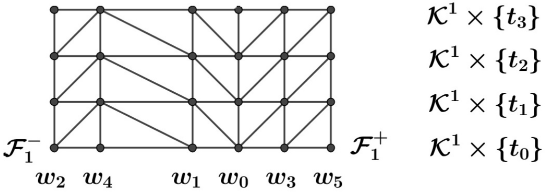

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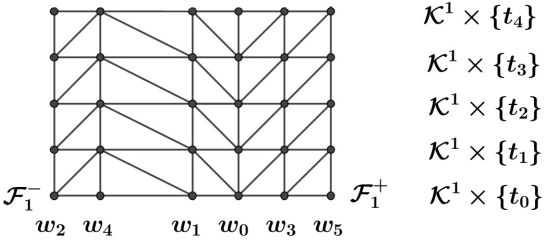

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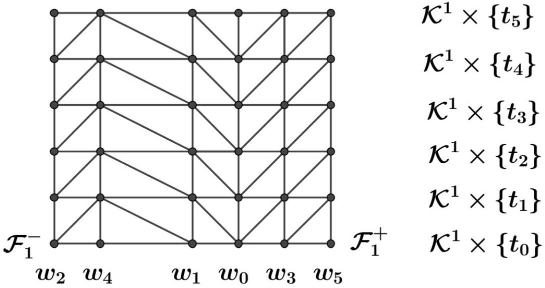

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Lemma (MK, Tkacz 2015)

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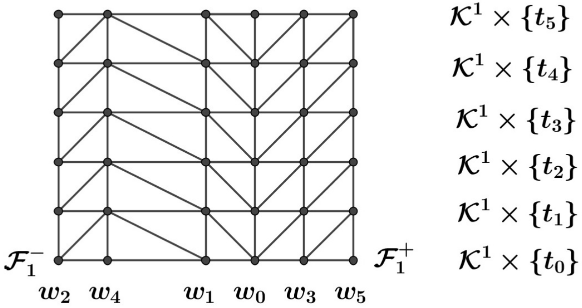

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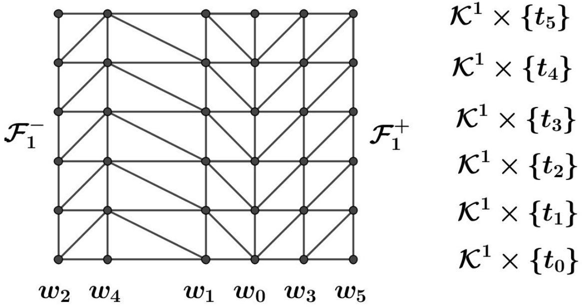

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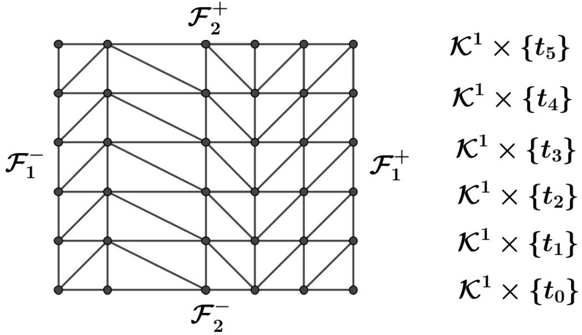

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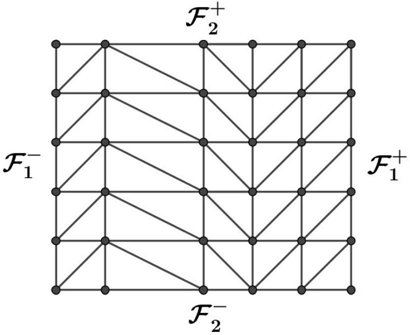

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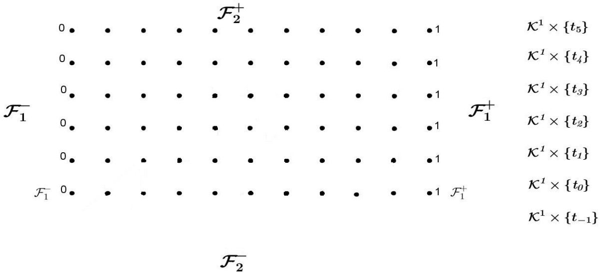

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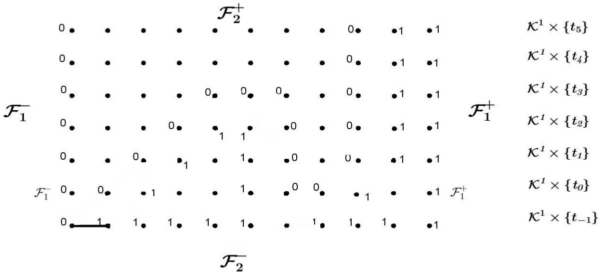

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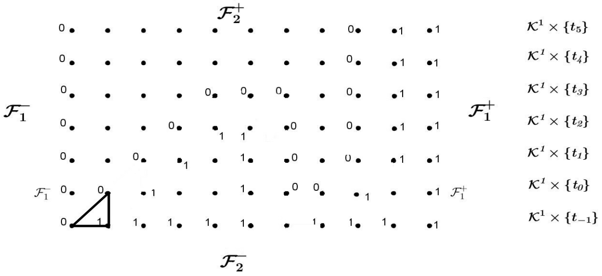

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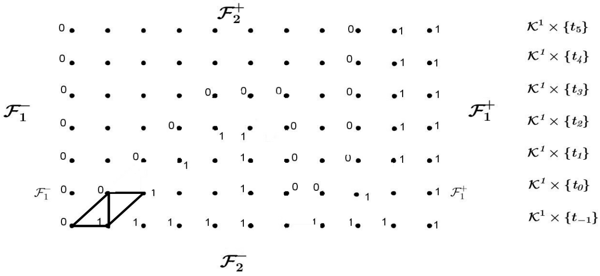

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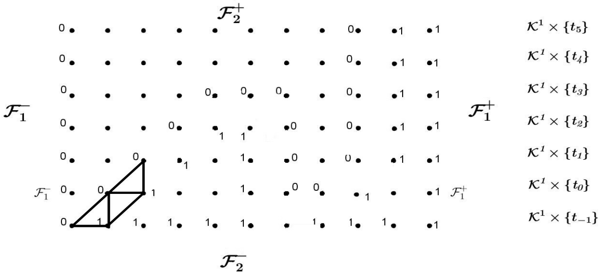

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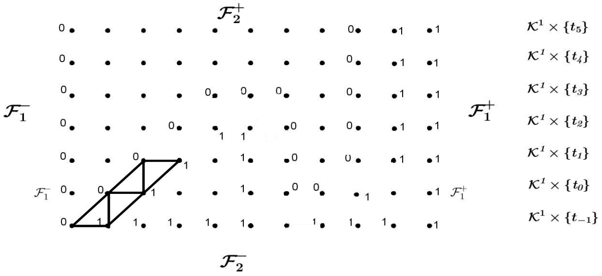

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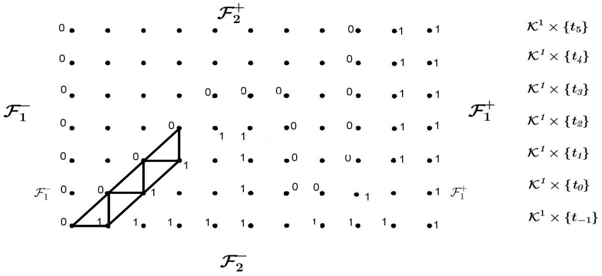

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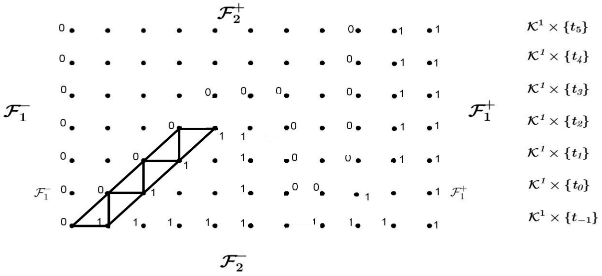

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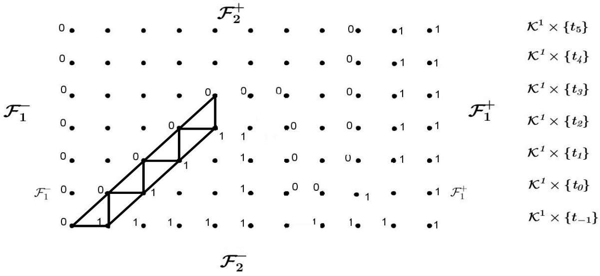

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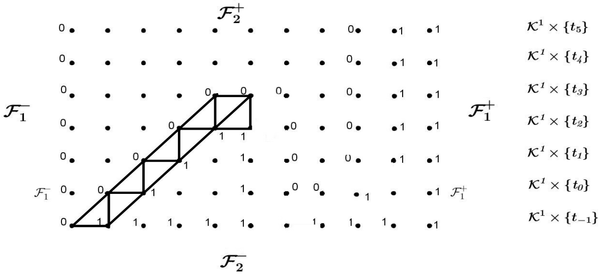

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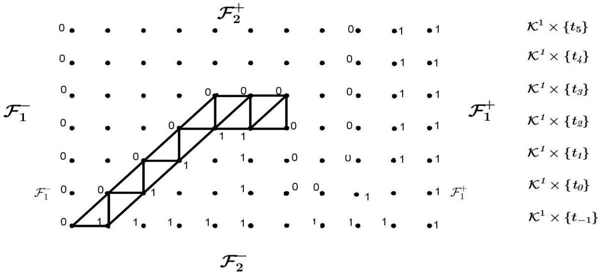

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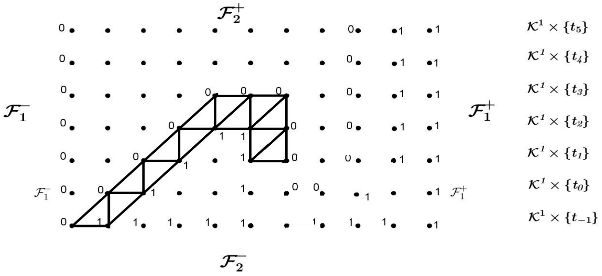

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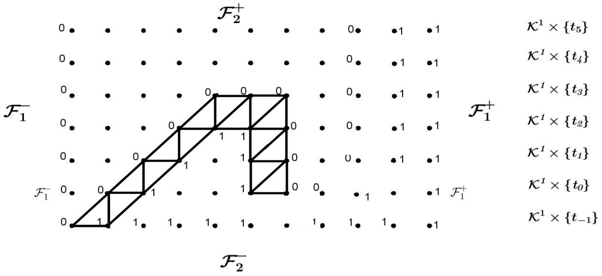

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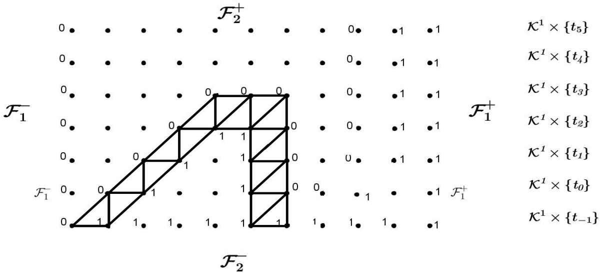

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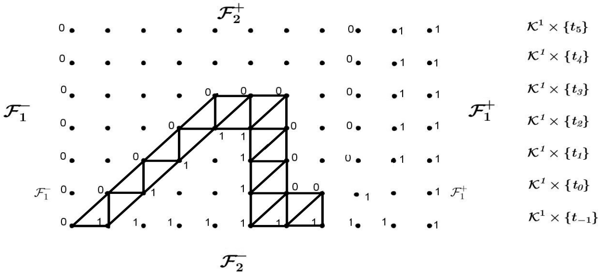

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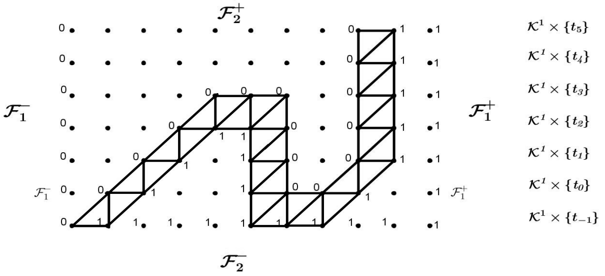

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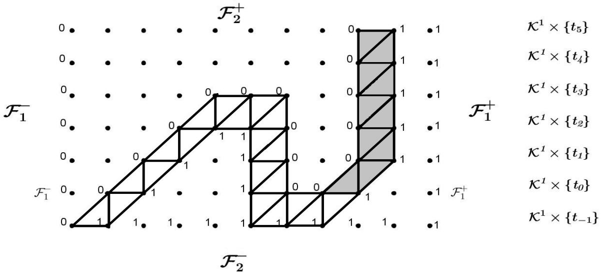

A topological part

Definition

The polyherdron in R^m is said to be an *n-cube-like polyhedron*, if its vertex-scheme abstract complex is an n-cube-like complex.

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Theorem (MK, Tkacz 2015)

Let $\{(H_i^-, H_i^+) : i \in \{1, \ldots, n\}\}$ *be a family of pairs of closed sets s. t.*

$$|\widetilde{\mathcal{F}}_i^-| \subset H_i^-, \ |\widetilde{\mathcal{F}}_i^+| \subset H_i^+ \ \ \text{and} \ \ |\widetilde{\mathcal{K}}^n \overset{\circ}{\times} L| = H_i^- \cup H_i^+,$$

then there exists a continuum $W \subset \bigcap_{i=1}^{n} H_i^- \cap H_i^+$ *with*

$$W \cap |\widetilde{\mathcal{K}}^n| \times \{t_0\} \neq \emptyset \neq W \cap |\widetilde{\mathcal{K}}^n| \times \{t_l\}.$$

An extension of the Poincaré-Miranda theorem

Theorem (MK, Tkacz 2015)

Let $(|\widetilde{\mathcal{K}}^n|, \widetilde{\mathcal{K}}^n)$ *be an n-cube-like polyhedron in* R^m

$$f = (f_1, \dots, f_n) \colon |\widetilde{\mathcal{K}}^n| \to R^n \text{ such that}$$

$$\forall_{i \leqslant n}\ f_i(|\mathcal{F}_i^-|) \subset (-\infty, 0],\ \ f_i(|\mathcal{F}_i^+|) \subset [0, \infty).$$

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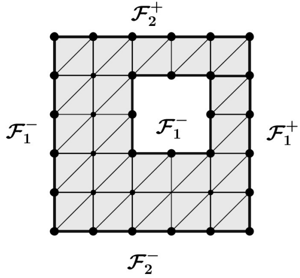

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Theorem (A parametric version; MK, Tkacz 2015)

Let $(|\widetilde{\mathcal{K}}^n|, \widetilde{\mathcal{K}}^n)$ *be an n-cube-like polyhedron in* R^m, $L = \{t_0, \ldots, t_l\} \subset R^k$,

$$f = (f_1, \ldots, f_n) \colon |\widetilde{\mathcal{K}}^n \overset{\circ}{\times} L| \to R^n \text{ such that}$$

$$\forall_{i \leqslant n}\ f_i(|\widetilde{\mathcal{F}}_i^-|) \subset (-\infty, 0],\ \ f_i(|\widetilde{\mathcal{F}}_i^+|) \subset [0, \infty).$$

Then there exists a continuum $W \subset f^{-1}(0)$ *with*

$$W \cap (|\widetilde{\mathcal{K}}^n| \times \{t_0\}) \neq \emptyset \neq W \cap (|\widetilde{\mathcal{K}}^n| \times \{t_l\}).$$