On discretely generated box products

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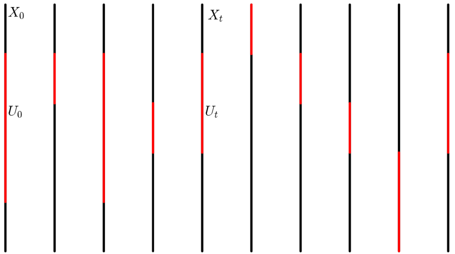
Definition

Let $\{X_t : t \in T\}$ be a family of topological spaces. The box product $\square_{t\in T} X_t$ of the spaces X_t is the set $\prod_{t\in T} X_t$ with the topology $\tau_\square$ generated by

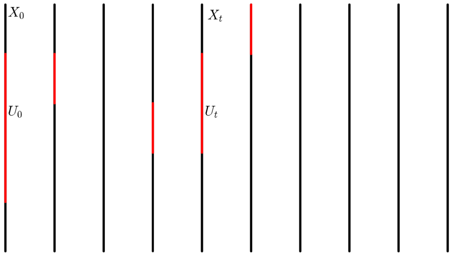
$$\{\prod_{t\in T} U_t : U_t \subseteq X_t \text{ open}\}.$$



Definition

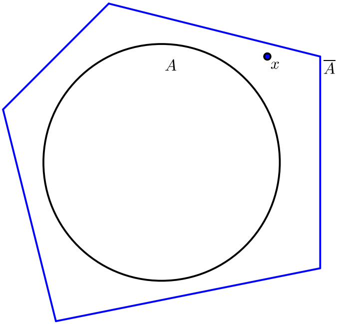
The Tychonoff product of the spaces X_t is the set $\prod_{t\in T} X_t$ with the topology τ generated by

$$\{\prod_{t\in F} U_t \times \prod_{t\in T\setminus F} X_t : U_t \subseteq X_t \text{ open } \wedge F \subseteq T \text{ finite}\}.$$



Definition (A. Dow, V. Tkachuk, M. Tkachenko and R. Wilson, 2002)

A topological space X is discretely generated if for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a discrete set $D \subseteq A$ such that $x \in \overline{D}$.



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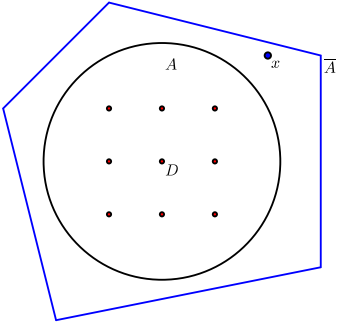

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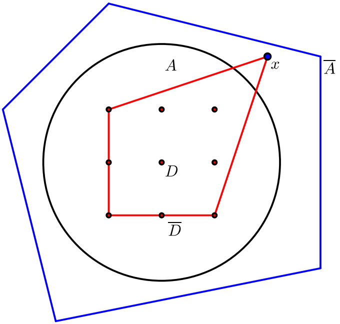

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- Does exist a NON discretely generated space? Yes. There is a countable regular space $\mathcal{V}$ that is not discretely generated (due to Eric van Douwen, 1993).

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- If the space 2^{ω_1} is not discretely generated then every dyadic compact discretely generated space is metrizable.
- If there exists an L-space then 2^{ω_1} is not discretely generated.
  (J. Moore proved the existance of an L-space in ZFC, 2006)

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- Is the Tychonoff product or the box product of $(\{\xi\} \cup \omega)^\omega$ discretely generated?

Relating box products

Theorem (Tkachuk-Wilson, 2012)

Consider a family of monotonically normal spaces $X_t, t \in T$. Then the box product $\square_{t\in T} X_t$ is discretely generated.

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- If X is first countable, is $\square X^\omega$ discretely generated?

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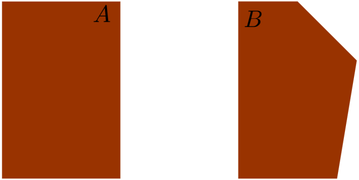

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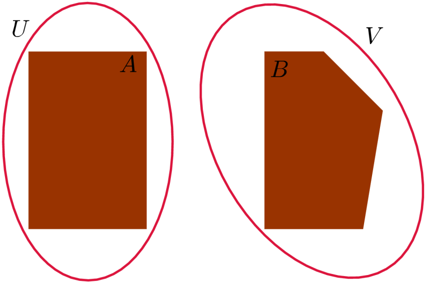

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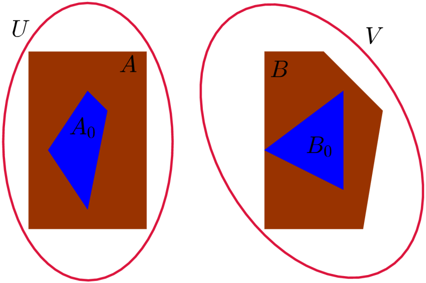

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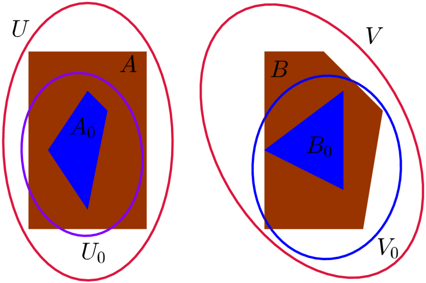

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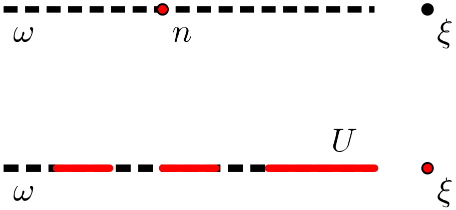
Topology on space $\{\xi\} \cup \omega$ is: $(U \in \xi)$



Motivation

Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines, for any $\xi \in \beta\omega \setminus \omega$?

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The answer is negative.

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Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines ($\square\mathbb{R}^\kappa$), for any $\xi \in \beta\omega \setminus \omega$? We want to show NO.

- Suppose there is an embedding $\varphi : \{\xi\} \cup \omega \to \square(\omega+1)^\omega$.

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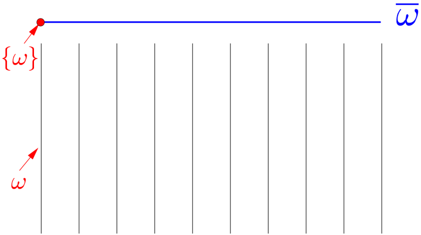
- Let $A = \varphi(\omega)$ and for simplicity suppose that $\varphi(\xi) = \overline{\omega}$. (the ceiling $\overline{\omega}$ is the constant function equal to ω)

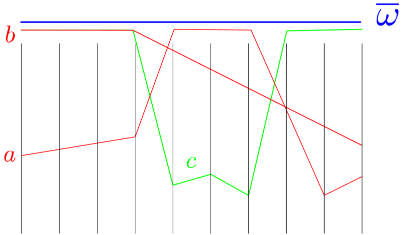


Sketch

- Denote $supp(a) = \{n \in \omega : a(n) \neq \omega\}$ and divide A as $A = A_\infty \cup A_0$, where

$$A_\infty = \{a \in A : supp(a) \ \text{is infinite}\} \ \text{ and}$$

$$A_0 = \{a \in A : supp(a) \ \text{is finite}\}.$$



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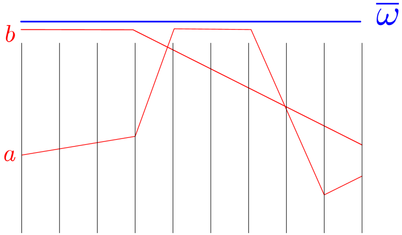
- We can ignore A_∞ because $|A_\infty| \leq \aleph_0 < \mathfrak{b}$, so is not enough to reach $\overline{\omega}$.



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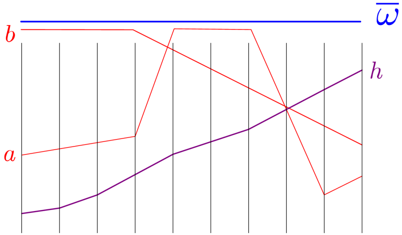

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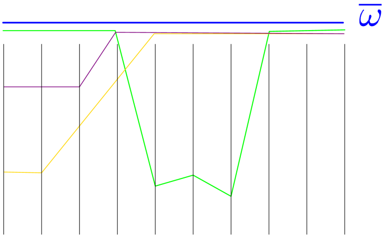
- Thus, assume $A = A_0$. Next, let

$$A = \bigsqcup_{F \in [\omega]^{<\omega}} A_F,$$

where $A_F = \{a \in A : supp(a) = F\}$.



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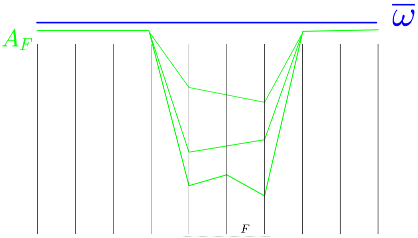

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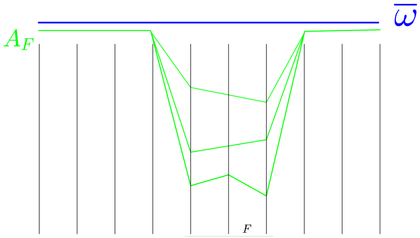
Two cases:

1. Suppose that $\overline{\omega} \in \overline{A_F}$, for some F.



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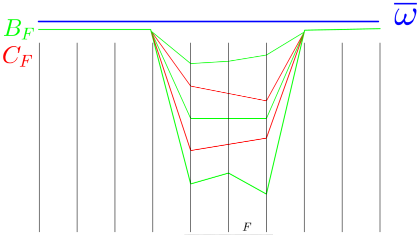

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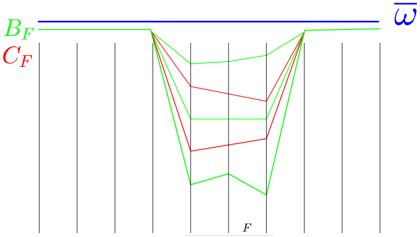
2. Suppose that $\overline{\omega} \notin \overline{A_F}$, for any F.



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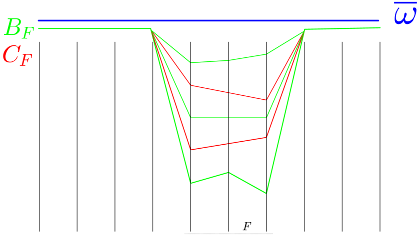

- The sets $B = \bigcup_{F \in [\omega]^{<\omega}} B_F$ and $C = \bigcup_{F \in [\omega]^{<\omega}} C_F$ are disjoint subsets of A and $\overline{\omega} \in \overline{B} \cap \overline{C}$.

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- The sets $U = \varphi^{-1}[B]$ and $V = \varphi^{-1}[C]$ work for the contradiction!

Na zdraví for your attention!