C^*-algebras and B-names for Complex Numbers

Andrea Vaccaro

University of Pisa

Hejnice, Winter School 2016

Summary

1. C^*-algebras
2. C^*-algebras with extremely disconnected spectrum
3. Boolean Valued Models
4. B-names for Complex Numbers

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\| \rangle$ is a Banach space
2. $\|xz\| \leq \|x\|\|z\|$
3. $(x + y)^* = x^* + y^*$
4. $(xy)^* = y^* x^*$
5. $(\lambda x)^* = \overline{\lambda} x^*$
6. $x^{**} = x$
7. $\|x^* x\| = \|x\|^2$

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\| \rangle$ is a Banach space
2. $\|xz\| \leq \|x\|\|z\|$
3. $(x+y)^* = x^* + y^*$
4. $(xy)^* = y^* x^*$
5. $(\lambda x)^* = \overline{\lambda} x^*$
6. $x^{**} = x$
7. $\|x^* x\| = \|x\|^2$

Example

- Commutative: $L^\infty([0,1])$, $\mathcal{C}(X)$ for X compact Hausdorff

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\| \rangle$ is a Banach space
2. $\|xz\| \leq \|x\|\|z\|$
3. $(x+y)^* = x^* + y^*$
4. $(xy)^* = y^* x^*$
5. $(\lambda x)^* = \overline{\lambda} x^*$
6. $x^{**} = x$
7. $\|x^* x\| = \|x\|^2$

Example

- Commutative: $L^\infty([0,1])$, $\mathcal{C}(X)$ for X compact Hausdorff
- Non-commutative: $\mathcal{B}(H)$ for H Hilbert space

The Gelfand-Naimark Theorem

Definition (Spectrum)

The **spectrum** of $\mathcal{A}$ is the set

$$\sigma(\mathcal{A}) = \{h \in \mathcal{A}^* : h(e) = 1 \wedge h(xy) = h(x)h(y)\}$$

The Gelfand-Naimark Theorem

Definition (Spectrum)

The **spectrum** of $\mathcal{A}$ is the set

$$\sigma(\mathcal{A}) = \{h \in \mathcal{A}^* : h(e) = 1 \wedge h(xy) = h(x)h(y)\}$$

Proposition

The spectrum of a commutative unital C^*-algebra is an Hausdorff compact subspace of $\mathcal{A}^*$ in the weak* topology.

The Gelfand-Naimark Theorem

Definition (Spectrum)

The **spectrum** of $\mathcal{A}$ is the set

$$\sigma(\mathcal{A}) = \{h \in \mathcal{A}^* : h(e) = 1 \wedge h(xy) = h(x)h(y)\}$$

Proposition

The spectrum of a commutative unital C^*-algebra is an Hausdorff compact subspace of $\mathcal{A}^*$ in the weak* topology.

Theorem (Gelfand-Naimark)

Assume $\mathcal{A}$ is a commutative and unital C^*-algebra. Then

$$\mathcal{A} \cong \mathcal{C}(\sigma(\mathcal{A}))$$

The Space $\mathcal{C}(St(\mathrm{B}))$

The Space $\mathcal{C}(St(\mathsf{B}))$

Theorem

Let $\mathcal{A}$ be a commutative unital C^*-algebra, $X = \sigma(\mathcal{A})$ its spectrum and $\mathsf{B} = \mathsf{RO}(X)$. Assume furthermore that X is extremely disconnected ($\mathsf{RO}(X) = \mathsf{CL}(X)$). Then X is homeomorphic to $St(\mathsf{B})$ and there exists an isometric $*$-isomorphism of C^*-algebras between $\mathcal{C}(X)$ and $\mathcal{C}(St(\mathsf{B}))$.

The Space $\mathcal{C}(St(\mathsf{B}))$

Theorem

Let $\mathcal{A}$ be a commutative unital C^*-algebra, $X = \sigma(\mathcal{A})$ its spectrum and $\mathsf{B} = \mathsf{RO}(X)$. Assume furthermore that X is extremely disconnected ($\mathsf{RO}(X) = \mathsf{CL}(X)$). Then X is homeomorphic to $St(\mathsf{B})$ and there exists an isometric $*$-isomorphism of C^*-algebras between $\mathcal{C}(X)$ and $\mathcal{C}(St(\mathsf{B}))$.

Remark

The previous result and the Gelfand-Naimark Theorem tell us that:

$$\mathcal{A} \cong \mathcal{C}(X) \cong \mathcal{C}(St(\mathsf{B}))$$

for $\mathsf{B} = \mathsf{RO}(X) = \mathsf{CL}(X)$.

Is $\mathcal{C}(St(\mathrm{B}))$ an exotic space?

Is $\mathcal{C}(St(\mathrm{B}))$ an exotic space?

There are well understood examples of such spaces.

Is $\mathcal{C}(St(\mathsf{B}))$ an exotic space?

There are well understood examples of such spaces.
Let

$$\mathsf{B} = \mathsf{MALG}$$

Is $\mathcal{C}(St(\mathsf{B}))$ an exotic space?

There are well understood examples of such spaces.
Let

$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

Is $\mathcal{C}(St(\mathsf{B}))$ an exotic space?

There are well understood examples of such spaces.
Let

$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

This is a complete boolean algebra.

Is $\mathcal{C}(St(\mathsf{B}))$ an exotic space?

There are well understood examples of such spaces.
Let

$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

This is a complete boolean algebra.
Using Gelfand Transform it can be shown that:

Proposition

$$\mathcal{C}(St(\mathsf{MALG})) \cong L^{\infty}([0,1])$$

$\mathcal{C}(St(\mathrm{B}))$ is not enough

$\mathcal{C}(St(\mathrm{B}))$ is not enough

Definition

$\mathcal{C}^+(St(\mathrm{B}))$ is the set of continuous functions f from $St(\mathrm{B})$ to the one point compactification of the complex field $\mathbb{C} \cup \{\infty\} \cong \mathcal{S}^2$ such that $f^{-1}(\{\infty\})$ is nowhere dense.

$\mathcal{C}(St(\mathrm{B}))$ is not enough

Definition

$\mathcal{C}^+(St(\mathrm{B}))$ is the set of continuous functions f from $St(\mathrm{B})$ to the one point compactification of the complex field $\mathbb{C} \cup \{\infty\} \cong \mathcal{S}^2$ such that $f^{-1}(\{\infty\})$ is nowhere dense.

Example

Let $\mathrm{B} = \mathrm{MALG}$. We know that $L^\infty([0,1]) \cong \mathcal{C}(St(\mathrm{MALG}))$. The space $\mathcal{C}^+(St(\mathrm{MALG})) \cong L^{\infty+}([0,1])$ has $1/x$ (modulo isomorphism) among its elements.

Spoiler Alert

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix* G *a ultrafilter on it.*

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix G a ultrafilter on it. Assume $R_1, \ldots, R_n$ are Borel relations and $F_1, \ldots, F_m$ Borel functions on $\mathbb{C}$.*

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix G a ultrafilter on it. Assume $R_1, \ldots, R_n$ are Borel relations and $F_1, \ldots, F_m$ Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle L^{\infty+}([0,1])/G, R_1/G, \ldots, F_m/G \rangle$$

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix G a ultrafilter on it. Assume $R_1, \ldots, R_n$ are Borel relations and $F_1, \ldots, F_m$ Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle L^{\infty+}([0,1])/G, R_1/G, \ldots, F_m/G \rangle$$

Remark

- $L^{\infty+}([0,1])/G \cong \mathcal{C}^+(St(\text{MALG}))/G$ *is the ring of germs functions in $\mathcal{C}^+(St(\text{MALG}))$ at the point G:*

$$[f]_G = [g]_G \Leftrightarrow \exists a \in G \text{ such that } f \restriction_{\mathcal{O}_a} = g \restriction_{\mathcal{O}_a}$$

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix G a ultrafilter on it. Assume $R_1, \ldots, R_n$ are Borel relations and $F_1, \ldots, F_m$ Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle L^{\infty +}([0,1])/G, R_1/G, \ldots, F_m/G \rangle$$

Remark

- $L^{\infty +}([0,1])/G \cong \mathcal{C}^+(St(\mathrm{MALG}))/G$ *is the ring of germs functions in $\mathcal{C}^+(St(\mathrm{MALG}))$ at the point G:*

$$[f]_G = [g]_G \Leftrightarrow \exists a \in G \text{ such that } f \restriction_{\mathcal{O}_a} = g \restriction_{\mathcal{O}_a}$$

- *This is is an algebraically closed field which extends $\mathbb{C}$ and which preserves the truth value of Σ_2 formulae of $\mathbb{C}$.*

Spoiler Alert

Theorem

Consider the complete boolean algebra MALG *and fix G a ultrafilter on it. Assume $R_1, \dots, R_n$ are Borel relations and $F_1, \dots, F_m$ Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \dots, F_m \rangle \prec_{\Sigma_2} \langle L^{\infty +}([0,1])/G, R_1/G, \dots, F_m/G \rangle$$

Remark

- $L^{\infty +}([0,1])/G \cong \mathcal{C}^+(St(\mathrm{MALG}))/G$ *is the ring of germs functions in $\mathcal{C}^+(St(\mathrm{MALG}))$ at the point G:*

$$[f]_G = [g]_G \Leftrightarrow \exists a \in G \text{ such that } f \restriction_{\mathcal{O}_a} = g \restriction_{\mathcal{O}_a}$$

- *This is is an algebraically closed field which extends $\mathbb{C}$ and which preserves the truth value of Σ_2 formulae of $\mathbb{C}$.*
- *This is not the case for $L^{\infty}([0,1])/G$.*

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

An $\mathcal{L}$-**structure** is a tuple

$$\mathcal{M} = \langle M, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

An $\mathcal{L}$-**structure** is a tuple

$$\mathcal{M} = \langle M, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

where:

- M is a non-empty set;
- $R_i^{\mathcal{M}}$ is a subset of M^{n_i};
- $f_j^{\mathcal{M}}$ is a function from M^{m_j} to M;
- $c_k^{\mathcal{M}}$ is a element of M.

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

A B-valued model is a tuple

$$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

A B-valued model is a tuple

$$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

where:

- M is a non-empty set;
- $R_i^{\mathcal{M}}$ is a function:

$$R_i^{\mathcal{M}} : M^{n_i} \to \mathsf{B}$$
$$(\tau_1, \ldots, \tau_{n_i}) \mapsto [\![R_i(\tau_1, \ldots, \tau_{n_i})]\!]_{\mathsf{B}}^{\mathcal{M}}$$

- $f_j^{\mathcal{M}}$ is a function:

$$f_j^{\mathcal{M}} : M^{m_j+1} \to \mathsf{B}$$
$$(\tau_1, \ldots, \tau_{m_j}, \sigma) \mapsto \left[\!\!\left[f_j(\tau_1, \ldots, \tau_{m_j}) = \sigma \right]\!\!\right]_{\mathsf{B}}^{\mathcal{M}}$$

- $c_k^{\mathcal{M}}$ is a element of M.

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\psi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg\, [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\exists x \phi(x)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]^{\mathcal{M}}_{\mathsf{B}}$

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\psi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg\, [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\exists x \phi(x)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]^{\mathcal{M}}_{\mathsf{B}}$

Remark

A B*-valued model* $\mathcal{M}$ *associates to each formula* ϕ *a value in* B*.*
First order models are B*-valued model for* $\mathsf{B} = \{0, 1\}$*.*

Assume $\mathcal{M}$ is a B-valued model and $G \in St(\mathsf{B})$. The following:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

is an equivalence relation. The quotient $\mathcal{M}/G$ has a natural structure of first order model.

Assume $\mathcal{M}$ is a B-valued model and $G \in St(\mathsf{B})$. The following:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

is an equivalence relation. The quotient $\mathcal{M}/G$ has a natural structure of first order model.

Definition

A B-valued model $\mathcal{M}$ is **full** if for any formula $\phi(x)$ there exists $\tau \in M$ such that:

$$[\![\exists x \phi(x)]\!] = [\![\phi(\tau)]\!]$$

Assume $\mathcal{M}$ is a B-valued model and $G \in St(\mathsf{B})$. The following:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

is an equivalence relation. The quotient $\mathcal{M}/G$ has a natural structure of first order model.

Definition

A B-valued model $\mathcal{M}$ is **full** if for any formula $\phi(x)$ there exists $\tau \in M$ such that:

$$[\![\exists x \phi(x)]\!] = [\![\phi(\tau)]\!]$$

Theorem (Boolean Valued Models Łoś's Theorem)

Assume $\mathcal{M}$ is a full B*-valued model for the language $\mathcal{L}$. Let $G \in St(\mathsf{B})$. Then $\mathcal{M}/G$ is a first order model for $\mathcal{L}$ and for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

$$\mathcal{M}/G \models \phi([\tau_1]_G, \ldots [\tau_n]_G) \Leftrightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$$

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$. We define

$$[\![R(f,g)]\!] = \mathring{\overline{\{G \in St(\mathsf{B}) : f(G) R g(G)\}}} = \mathring{\overline{(f \times g)^{-1}[R]}}$$

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. We define

$$[\![R(f,g)]\!] = \overset{\circ}{\overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}} = \overset{\circ}{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. We define

$$[\![R(f,g)]\!] = \overset{\circ}{\overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}} = \overset{\circ}{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

- $\mathcal{C}(St(\mathrm{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$. We define

$$[\![R(f,g)]\!] = \overset{\circ}{\overline{\{G \in St(\mathsf{B}) : f(G) R g(G)\}}} = \overset{\circ}{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B (B $\cong$ RO(St(B))).

- $\mathcal{C}(St(\mathsf{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathsf{B}))$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. We define

$$[\![R(f,g)]\!] = \overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}^{\circ} = \overline{(f \times g)^{-1}[R]}^{\circ}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

- $\mathcal{C}(St(\mathrm{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathrm{B}))$.

Proposition

$\mathcal{C}(St(\mathrm{B}))$ *is a* B*-valued extension of* $\mathbb{C}$*.*

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$. We define

$$[\![R(f,g)]\!] = \overset{\circ}{\overline{\{G \in St(\mathsf{B}) : f(G) R g(G)\}}} = \overset{\circ}{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathsf{B} \cong \mathrm{RO}(St(\mathsf{B}))$).

- $\mathcal{C}(St(\mathsf{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathsf{B}))$.

Proposition

$\mathcal{C}(St(\mathsf{B}))$ is a B-valued extension of $\mathbb{C}$. In particular, $L^\infty([0,1])$ is a MALG-valued extension of $\mathbb{C}$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$. We define

$$[\![R(f,g)]\!] = \mathring{\overline{\{G \in St(\mathsf{B}) : f(G)Rg(G)\}}} = \mathring{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathsf{B} \cong \mathrm{RO}(St(\mathsf{B}))$).

- $\mathcal{C}(St(\mathsf{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathsf{B}))$.

Proposition

$\mathcal{C}(St(\mathsf{B}))$ is a B-valued extension of $\mathbb{C}$. In particular, $L^\infty([0,1])$ is a MALG-valued extension of $\mathbb{C}$.

Remark

$\mathcal{C}(St(\mathsf{B}))$ is not full. $\mathcal{C}^+(St(\mathsf{B}))$ is full.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra. We construct V^{B} generalizing the construction of V.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra. We construct V^{B} generalizing the construction of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra. We construct V^{B} generalizing the construction of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.
$\mathcal{P}_{\mathsf{B}}(X) \equiv \mathsf{B}^X$ where for $f : X \to \mathsf{B}$, $f(a)$ is the boolean value of the concept *a belongs to X*.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra. We construct V^{B} generalizing the construction of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.
$\mathcal{P}_{\mathsf{B}}(X) \equiv \mathsf{B}^X$ where for $f : X \to \mathsf{B}$, $f(a)$ is the boolean value of the concept *a belongs to X*.
Therefore:

$$V_0^{\mathsf{B}} = \emptyset$$

$$V_{\alpha+1}^{\mathsf{B}} = \left\{ f : X \to \mathsf{B} \mid X \subset V_\alpha^{\mathsf{B}} \right\}$$

$$V_\beta^{\mathsf{B}} = \bigcup_{\alpha<\beta} V_\alpha^{\mathsf{B}} \text{ if } \beta \text{ is a limit ordinal}$$

$$V^{\mathsf{B}} = \bigcup_{\alpha \in ON} V_\alpha^{\mathsf{B}}$$

Cohen's Forcing Theorem

Theorem (Cohen's Forcing Theorem)

Assume $\mathsf{B} \in V$ *is a complete boolean algebra and* $G \in St(\mathsf{B})$*. Then*

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \mathsf{ZFC}$$

Moreover

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \Leftrightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$$

Cohen's Forcing Theorem

Theorem (Cohen's Forcing Theorem)

Assume $\mathsf{B} \in V$ *is a complete boolean algebra and* $G \in St(\mathsf{B})$. *Then*

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \mathsf{ZFC}$$

Moreover

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \Leftrightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$$

Remark

Forcing is a machine which produces first order models of ZFC. *The truth value of undecidable formulae in these models depends on the combinatorial properties of* B *and on the choice of* G.

V^{B}/G is generally a model which extends properly V.

V^{B}/G is generally a model which extends properly V.

This means that:

$$\mathbb{C}^{V} \subsetneq \mathbb{C}^{V^{\mathsf{B}}/G}$$

V^{B}/G is generally a model which extends properly V.

This means that:

$$\mathbb{C}^{V} \subsetneq \mathbb{C}^{V^{\mathsf{B}}/G}$$

We are interested in those B-names in V^{B} which become complex numbers after we take the quotient of V^{B} by a ultrafilter.

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

We denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

We denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

$\mathcal{C}^+(St(\mathsf{B}))$ is a B-valued extension of $\mathbb{C}$

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

We denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

$\mathcal{C}^{+}(St(\mathsf{B}))$ is a B-valued extension of $\mathbb{C}$ $\longleftrightarrow$ $\mathbb{C}^{\mathsf{B}}$ is a B-valued extension of $\mathbb{C}$

$\mathcal{C}^+(St(\mathsf{B})) \cong \mathbb{C}^{\mathsf{B}}$

Theorem

Fix a set

$$\mathcal{L} = \{R_i : i \in I\} \cup \{F_j : j \in J\}$$

where:

- *for* $i \in I$, R_i *is a Borel subset of* $\mathbb{C}^{n_i}$*;*
- *for* $j \in J$, F_j *is a Borel function from* $\mathbb{C}^{m_j}$ *to* $\mathbb{C}$*.*

Then

$$\mathcal{C}^+(St(\mathsf{B})) \cong \mathbb{C}^{\mathsf{B}}$$

as B*-valued models in the language* $\mathcal{L}$*.*

Generic Absoluteness

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$.

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$*. Assume* $R_1, \dots, R_n$ *are Borel relations on* $\mathbb{C}$ *and* $F_1, \dots, F_m$ *are Borel functions on* $\mathbb{C}$*.*

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$*. Assume* $R_1, \ldots, R_n$ *are Borel relations on* $\mathbb{C}$ *and* $F_1, \ldots, F_m$ *are Borel functions on* $\mathbb{C}$*. Then:*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathsf{B}}/G, R_1/G, \ldots, F_m/G \rangle$$

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$. *Assume* $R_1, \ldots, R_n$ *are Borel relations on* $\mathbb{C}$ *and* $F_1, \ldots, F_m$ *are Borel functions on* $\mathbb{C}$. *Then:*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathsf{B}}/G, R_1/G, \ldots, F_m/G \rangle$$

Therefore:

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathcal{C}^{+}(St(\mathsf{B}))/G, R_1/G, \ldots, F_m/G \rangle$$

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathrm{B})$. *Assume* $R_1, \ldots, R_n$ *are Borel relations on* $\mathbb{C}$ *and* $F_1, \ldots, F_m$ *are Borel functions on* $\mathbb{C}$. *Then:*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathrm{B}}/G, R_1/G, \ldots, F_m/G \rangle$$

Therefore:

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathcal{C}^+(St(\mathrm{B}))/G, R_1/G, \ldots, F_m/G \rangle$$

Remark

This means that we can use forcing as a mean to prove theorems within ZFC. *To prove that a* Σ^1_2*-formula* ϕ *is true in ZFC, it is not necessary to show that it holds in every model of ZFC. It is enough to find one model of a certain form in which* ϕ *holds.*

Back to C^*-algebras

Definition

Consider B a complete boolean algebra and let $\mathrm{B}^+ = \mathrm{B} \setminus \{0_\mathrm{B}\}$
$D \subseteq \mathrm{B}^+$ is **dense** if for each $b \in \mathrm{B}^+$ there exists $d \in D$ such that $d \leq b$.
$G \subseteq \mathrm{B}^+$ is **generic** over a class C (or C-**generic**) if:

- G is a filter;
- if $D \subseteq \mathrm{B}^+$ is dense and $D \in C$, then $G \cap D \neq \emptyset$.

Back to C^*-algebras

Definition

Consider B a complete boolean algebra and let $\mathsf{B}^+ = \mathsf{B} \setminus \{0_\mathsf{B}\}$
$D \subseteq \mathsf{B}^+$ is **dense** if for each $b \in \mathsf{B}^+$ there exists $d \in D$ such that $d \leq b$.
$G \subseteq \mathsf{B}^+$ is **generic** over a class C (or C-**generic**) if:

- G is a filter;
- if $D \subseteq \mathsf{B}^+$ is dense and $D \in C$, then $G \cap D \neq \emptyset$.

Proposition

Assume G is a V-generic filter on B. Then

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathcal{C}(St(\mathsf{B}))/G$$

$\mathcal{C}(St(\mathrm{B}))$ is enough!

Theorem

Let V be a transitive model of ZFC, $\mathrm{B} \in V$ *which V models to be a complete boolean algebra, and G a V-generic filter in* B. *Assume $R_1, \ldots, R_n$ are Borel relations and $F_1, \ldots, F_m$ Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathcal{C}(St(\mathrm{B}))/G, R_1/G, \ldots, F_m/G \rangle$$

Thanks for your attention!

Essential bibliography

[1] Andrea Vaccaro and Matteo Viale, *C*-algebras and B-names for complex numbers* (2015). Preprint, https://www.newton.ac.uk/files/preprints/ni15089.pdf.

[2] Matteo Viale, *Martin's maximum revisited* (2012). (http://arxiv.org/abs/1110.1181).

[3] ———, *Forcing the truth of a weak form of Schanuel's conjecture* (2015). Preprint, https://www.newton.ac.uk/files/preprints/ni15087.pdf.