Local function vs. local closure function

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Ideal topological space

Let τ be a topology on X. Then

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$A^*_{(\tau,\mathcal{I})}$ (briefly A^*) is called the **local function**
$\langle X, \tau, \mathcal{I} \rangle$ is an **ideal topological space** [Kuratowski 1933].

More on local function

(1) $A \subseteq B \Rightarrow A^* \subseteq B^*$;
(2) $A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$;
(3) $(A^*)^* \subseteq A^*$;
(4) $(A \cup B)^* = A^* \cup B^*$.
(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

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is a closure operator on $P(X)$ and it generates a topology $\tau^*(\mathcal{I})$ (briefly τ^*) on X where

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$$\tau \subseteq \tau^* \subseteq P(X)$$

Some results in ideal topological space

Theorem

For $\langle \mathbb{R}, \tau_{nat} \rangle$, and ideal of sets with Lebesgue measure 0, τ^*_{nat}-Borel sets are Lebesgue-measurable sets in τ_{nat} [Scheinberg, 1971]

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Theorem

Generalization of Cantor-Bendixson theorem: For an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$, where $\mathcal{I}$ is compatible with τ and contains *Fin*, τ^*-closed sets are union of a perfect set in τ and a set from the $\mathcal{I}$. [Freud, 1958]

θ-closure

U is a θ-open [Veličko 1966] iff for every $x \in U$ exists $V \in \tau(x)$ s.t. $\mathrm{Cl}(V) \subseteq U$

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$\langle X, \tau \rangle$ is T_3: open $\Rightarrow$ θ-open, $\tau = \tau_\theta$

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Space is T_2 iff every compact set is θ-closed. [Janković, 1980]

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Space is *H*-closed if every open cover has a finite subfamily such that their closures cover it

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Theorem

Every H-closed space with ccc is not a union of less than continuum many θ-closed nowhere dense sets if and only if Martin's axiom holds. [Dickman and Porter, 1975]

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Combining these two we get

$$\Gamma_{(\tau,\mathcal{I})}(A) = \{x \in X : A \cap \mathrm{Cl}(U) \notin \mathcal{I} \text{ for each } U \in \tau(x)\}.$$

$\Gamma_{(\tau,\mathcal{I})}(A)$ (briefly $\Gamma(A)$) is **local closure function** [Al-Omari, Noiri 2014]

$\Gamma(A)$ and $\psi_\Gamma(A)$

(1) $A^* \subseteq \Gamma(A)$;
(2) $\Gamma(A) = \mathrm{Cl}(\Gamma(A)) \subseteq \mathrm{Cl}_\theta(A)$;
(3) $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B)$;
(4) $\Gamma(A \cup I) = \Gamma(A) = \Gamma(A \setminus I)$ for each $I \in \mathcal{I}$.

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$\psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$ [Al-Omari, Noiri 2014]

(1) $\psi_\Gamma(A) = \mathrm{Int}(\psi_\Gamma(A))$;
(2) $\psi_\Gamma(A \cap B) = \psi_\Gamma(A) \cap \psi_\Gamma(B)$;
(3) $\psi_\Gamma(A \cup I) = \psi_\Gamma(A) = \psi_\Gamma(A \setminus I)$ for each $I \in \mathcal{I}$;
(4) If U is θ-open, then $U \subseteq \psi_\Gamma(U)$.

Ideals

$\langle X, \tau \rangle$ - topological space

Fin - ideal of finite sets
$\mathcal{I}_{ctble}$ - ideal of countable sets
$\mathcal{I}_{cd}$ - ideal of closed discrete sets.

S is scattered if each nonempty subset of S contains an isolated point.

$\mathcal{I}_{sc}$ - ideal of scattered sets (if X is T_1)

A is relatively compact if $\mathrm{Cl}(A)$ is compact.

$\mathcal{I}_K$ - ideal of relatively compact sets

A is nowhere dense if $\mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$

$\mathcal{I}_{nwd}$ - ideal of nowhere dense sets

Countable union of nowhere dense sets is called a meager set

$\mathcal{I}_{mg}$ - ideal of meager sets

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$$\begin{array}{ccccccc} \tau_\theta & \subset & \tau & \subset & \tau^* & \subset & P(X) \\ \cap & & & & & & \\ \sigma & \subseteq & \sigma_0 & & & & \end{array}$$

On inequality of σ and σ_0

Question [Al-Omari, Noiri 2014]: $\sigma \subsetneq \sigma_0$?

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Lemma

If $\sigma \subsetneq \sigma_0$, then there exists a set A and a point $x \in A$ such that:
(1) $\mathrm{Cl}(U) \setminus A \notin \mathcal{I}$, for each $U \in \tau(x)$, and
(2) there exist $V \in \tau(x)$ and an open set $W \subseteq V$ such that $\mathrm{Cl}(W) \setminus A \in \mathcal{I}$.

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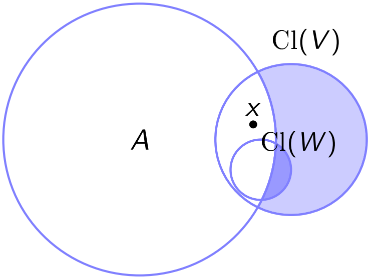

On inequality of σ and σ_0

Example

$X = \omega \cup \{\omega\}$; $\tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}$; $\mathcal{I} = Fin$.

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Lemma conditions are fulfilled

Each open neighborhood of the point ω has the form $U = \{\omega\} \cup (\omega \setminus K)$, and $\mathrm{Cl}(U) \setminus \{\omega\} = \omega \setminus K \notin Fin$. But there exists $n_0 \in U$, so $\{n_0\}$ is a clopen singleton, such that $\mathrm{Cl}(\{n_0\}) \setminus A = \{n_0\} \in Fin$.

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$$\{\omega\} \notin \sigma.$$

$$\psi_\Gamma(\{\omega\}) = \omega.$$

The point ω is the only point with infinite closure of each its neighborhood. Therefore, it is not difficult to see that $\Gamma(\omega) = \{\omega\}$.

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$$\sigma \subset \sigma_0$$

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Theorem

Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then each of the following conditions implies that $\Gamma(A) = A^*$, for each set A.

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Theorem

Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then each of the following conditions implies that $\Gamma(A) = A^*$, for each set A.

a) The topology τ has a clopen base.

b) τ is a T_3-topology.

c) $\mathcal{I} = \mathcal{I}_{cd}$.

d) $\mathcal{I} = \mathcal{I}_K$.

e) $\mathcal{I}_{nwd} \subseteq \mathcal{I}$.

f) $\mathcal{I} = \mathcal{I}_{mg}$.

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Example

$X = \mathbb{R}$; $K = \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$; $\mathcal{I} = Fin$

$$\mathcal{B}(x) = \begin{cases} \{(x-a, x+a) : a > 0\}, & x \neq 0; \\ \{(-a, a) \setminus K : a > 0\}, & x = 0 \end{cases}$$

This neighbourhood system generates a T_2-topology which is not T_3 [Engelking, Example 1.5.6].

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Example

$K^* = \emptyset$.

For $x \neq 0$, there exists $U \in \mathcal{B}(x)$ such that $|U \cap K| \leq 1$, so $U \cap K \in Fin$, implying $x \notin K^*$. If $x = 0$, since $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$, we have $U \cap K = \emptyset$ for each $U \in \mathcal{B}(0)$, implying $0 \notin K^*$.

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$\Gamma(K) = \{0\}$

If $x \neq 0$, then there also exists $U \in \mathcal{B}(x)$ such that $|\operatorname{Cl}(U) \cap K| \leq 1$, so $\operatorname{Cl}(U) \cap K \in Fin$, implying $x \notin \Gamma(K)$. For $x = 0$ and $U \in \mathcal{B}(x)$ we have $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$. But $\operatorname{Cl}(U) = [-a, a]$, implying $|\operatorname{Cl}(U) \cap K| = \aleph_0$, so $\operatorname{Cl}(U) \cap K \notin Fin$.

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$K^* \subsetneq \Gamma(K)$

$\mathcal{I}_{ctble}$

Example

$X = \mathbb{R}$, $\mathcal{I} = \mathcal{I}_{ctble}$

$$\mathcal{B}(x) = \begin{cases} \{(x-a, x+a) \cap \mathbb{Q} : a \in \mathbb{R}^+\}, & x \in \mathbb{Q}; \\ \{(x-a, x+a) : a \in \mathbb{R}^+\}, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

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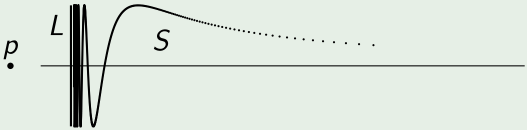
$\Gamma((-1,1)) = [-1,1]$

$\mathrm{Cl}((q-a, q+a) \cap \mathbb{Q}) = [q-a, q+a]$ for each $q \in \mathbb{Q}$, and its intersection with $[-1,1]$ is either empty, or a singleton, or a closed (uncountable) interval

$\mathcal{I}_{sc}$

Example

Let $S = \{\langle \frac{1}{n}, \sin n\rangle : n \in \mathbb{N}\} \subset \mathbb{R}^2$ and $L = \{0\} \times [-1, 1]$. Let $X = S \cup L \cup \{p\}$, where p is a special point outside of $\mathbb{R}^2$.



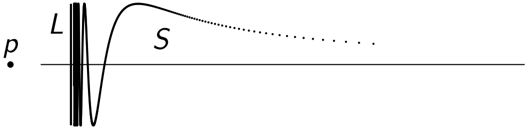
For $x \in S \cup L$ let $\mathcal{B}(x)$ be the neighbourhood system as in the induced topology on $S \cup L$ from $\mathbb{R}^2$

For the point p let $\mathcal{B}(p) = \{\{p\} \cup S \setminus K : K \in [S]^{<\aleph_0}\}$.

S is a scattered set.

$\mathcal{I} = \mathcal{I}_{sc}$ and $A = S \cup L$.

$\mathcal{I}_{sc}$



Example

$A^* = L$.

For $x \in S$, $\{x\} \cap A$ is a singleton, and therefore a scattered set.
For $x \in L$, each its neighbourhood contains an interval on the line L, so not scattered.
Each neighbourhood of p meets only S, so its intersection with A is scattered.

$\Gamma(A) = L \cup \{p\}$.

$L \subseteq \mathrm{Cl}(S)$.
$L \subseteq \mathrm{Cl}(S \setminus K)$, where K is finite.
For an open set $U = \{p\} \cup S \setminus K$, as a neighbourhood of p, we have $\mathrm{Cl}(U) = U \cup L$. So, $\mathrm{Cl}(U) \cap A$ contains L, which is dense in itself, and therefore $\mathrm{Cl}(U) \cap A$ is not scattered, implying $p \in \Gamma(A)$.
By the same reason as in the local function case, there is no point of S in $\Gamma(A)$.

So, $(S \cup L)^* \subsetneq \Gamma(S \cup L)$.

Fin, Nearly discrete spaces

We have seen for $\mathcal{I} = Fin$ that there exists an example such that $A^* \neq \Gamma(A)$.

the first example of T_2-space

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the first example of T_2-space

A topological space $\langle X, \tau \rangle$ is nearly discrete if each $x \in X$ has a finite neighbourhood.

Every nearly discrete space is an Alexandroff space (arbitrary intersection of open sets is open).

It is known that $X^*_{Fin} = \emptyset$ **if and only if** $\langle X, \tau \rangle$ is nearly discrete (see [Janković Hamlett 1990])

Fin, Nearly discrete spaces

We have seen for $\mathcal{I} = Fin$ that there exists an example such that $A^* \neq \Gamma(A)$.

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Theorem

For an ideal topological space $\langle X, \tau, Fin \rangle$, if $\Gamma(X) = \emptyset$, then $\langle X, \tau \rangle$ is nearly discrete.

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The converse is not true.

Example

Let $X = \omega$, $\mathcal{B} = \{\{0, i\} : i \in \omega\}$.

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$\{0\}$ is an open set and $\mathrm{Cl}(\{0\}) = \omega$.
$\Gamma(\omega) = \omega \neq \emptyset$.
Since $\mathrm{Cl}(\{0, i\}) \cap \omega = \omega \cap \omega = \omega \notin Fin$

Fin, θ-derived set

$A^{d_\omega} = \{x \in X : |A \cap U| \geq \aleph_0 \text{ for all } U \in \tau(x)\}$ is the set of all ω-accumulation points of the set A

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For T_1 spaces we have that the derived set (set of accumulation points)

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θ-derived set [Caldas, Jafari, Kovár 2004] is defined by

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Theorem

For the ideal topological space of the form $\langle X, \tau, Fin\rangle$ and each subset A of X in it we have $\Gamma(A) \subseteq D_\theta(A)$.

θ-derived set

Inclusion can be strict.

Example

Let us consider the left-ray topology on the real line, i.e., $\tau = \{(-\infty, a) : a \in \mathbb{R}\}$.

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