Products of Menger spaces

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joint work with Boaz Tsaban

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The Menger property

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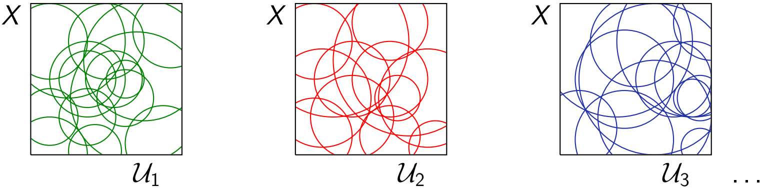

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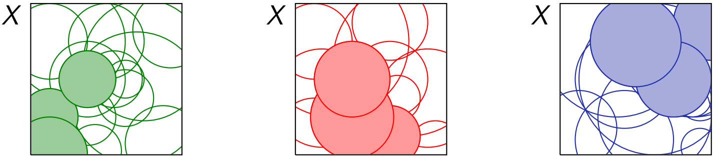

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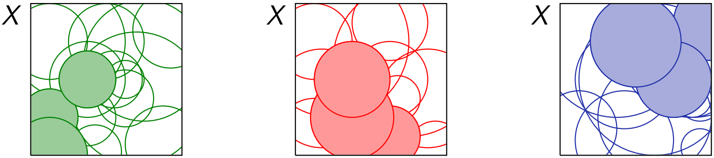

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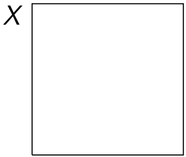

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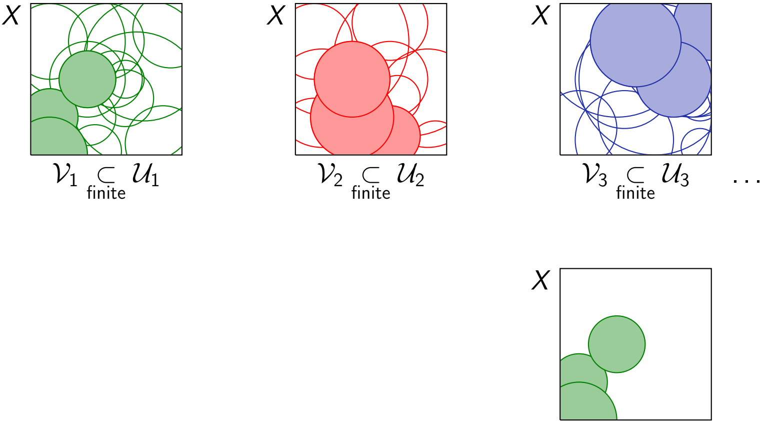

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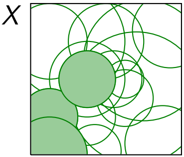

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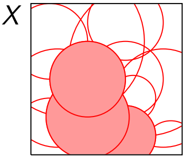

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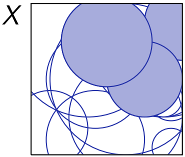

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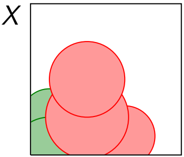

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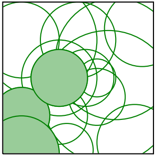
X



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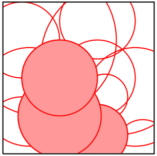

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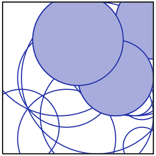

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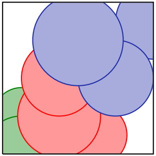

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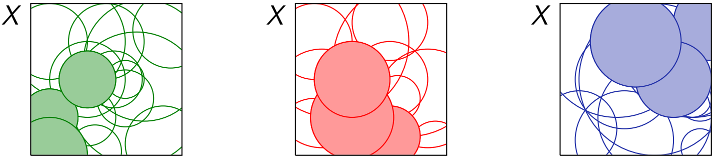

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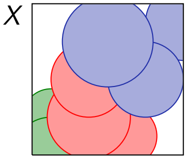
σ-compactness $\to$ Menger $\to$ Lindelöf



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$[\mathbb{N}]^\infty$: infinite subsets of $\mathbb{N}$

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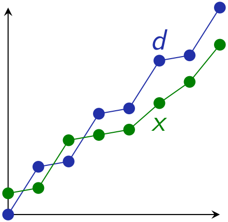

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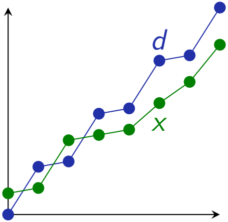

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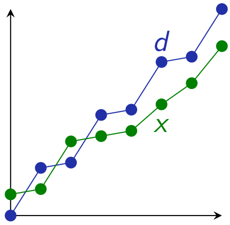

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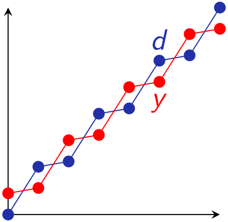

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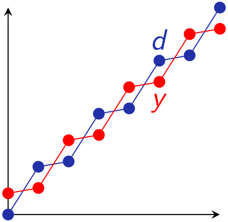

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Assume X is Lindelöf and zero-dimensional.
X is Menger iff $\forall \varphi : X \xrightarrow{cont} [\mathbb{N}]^\infty$, $\varphi[X]$ is non-dominating in $[\mathbb{N}]^\infty$

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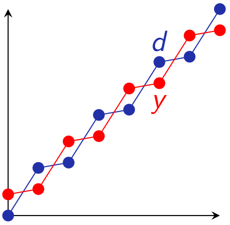

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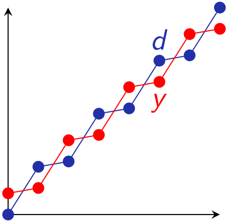

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- A dominating $X \subset [\mathbb{N}]^\infty$ is not Menger.

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Problem (Scheepers)

Is there (ZFC) a Menger set $M \subset \mathbb{R}$ such that M^2 is not Menger?

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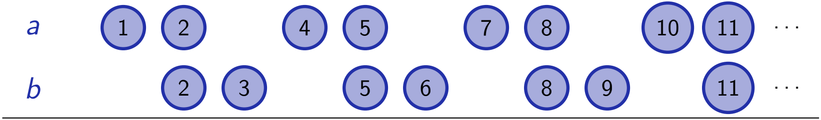

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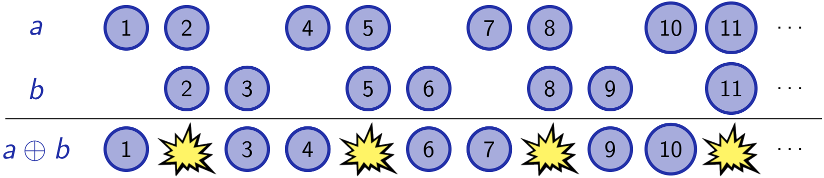

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b		2	3		5	6		8	9		11	$\cdots$
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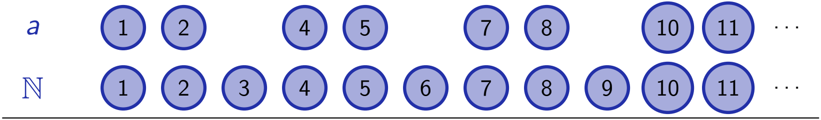

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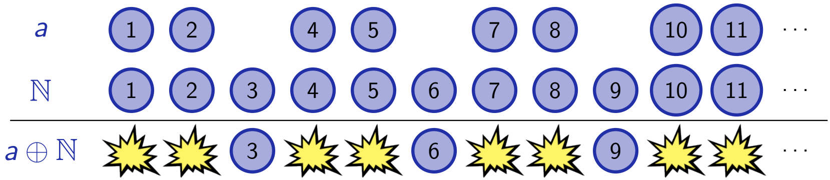

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$[\mathbb{N}]^{\infty}$: infinite subsets of $\mathbb{N}$

$[\mathbb{N}]^{\infty,\infty}$: infinite co-infinite subsets of $\mathbb{N}$

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if $|A| \geqslant \kappa$ and $\forall_{b \in [\mathbb{N}]^\infty} |\{ a \in A : a \leqslant^* b \}| < \kappa$

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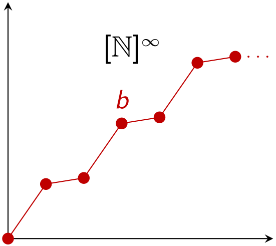

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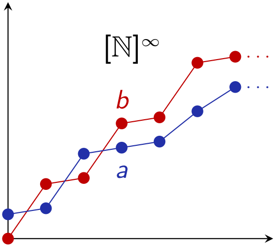

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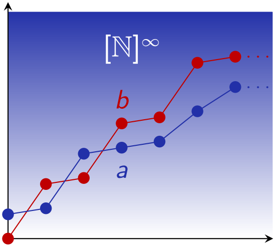

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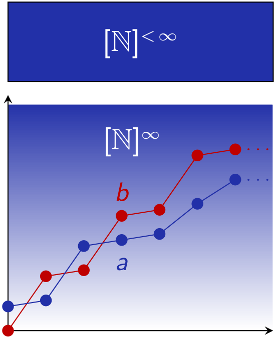

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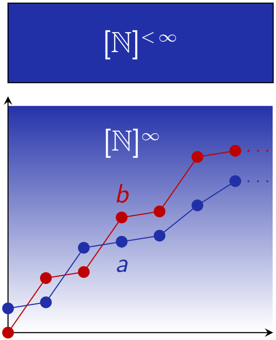

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If $X \subset [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbdd set or a $cf(\mathfrak{d})$-unbdd set, then there is a Menger $Y \subset P(\mathbb{N})$ such that $X \times Y$ is not Menger.

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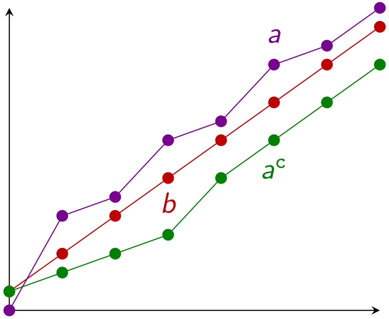

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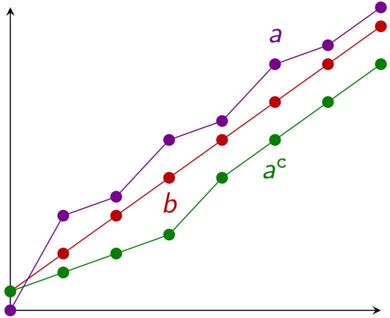

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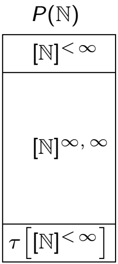

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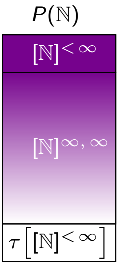

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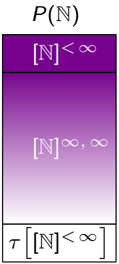

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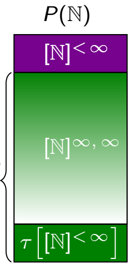

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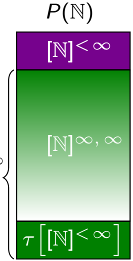

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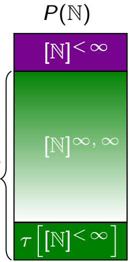

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$$|\mathbb{N}| = \aleph_0 \leqslant \mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{bidi} \leqslant \mathfrak{d} \leqslant \mathrm{cof}(\mathcal{M}) \leqslant \mathfrak{c} = |\mathbb{R}|$$

$$\max\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} \leqslant \mathfrak{bidi} \leqslant \min\{\mathfrak{r}, \mathfrak{d}\}$$

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Problem

Assume $\mathfrak{r} < \mathfrak{d}$ and $\mathfrak{d}$ is regular (e.g. in Miller's model).
Is there a Menger set $M \subset \mathbb{R}$ such that M^2 is not Menger?

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Another applications

Hurewicz's property: for every open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of X there are finite $\mathcal{V}_1 \subset \mathcal{U}_1, \mathcal{V}_2 \subset \mathcal{U}_2, \ldots$ such that $\{\, n \in \mathbb{N} : x \in \bigcup \mathcal{V}_n \,\}$ is co-finite for all $x \in X$.

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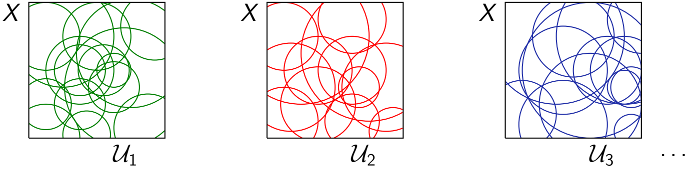

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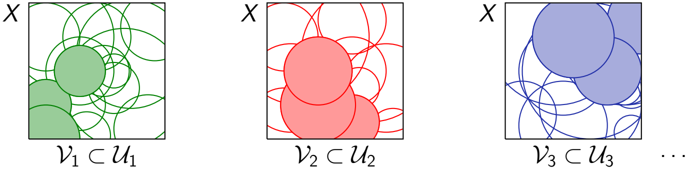

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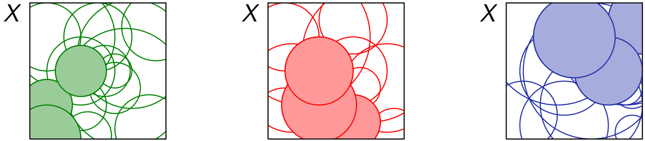

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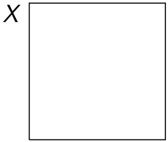

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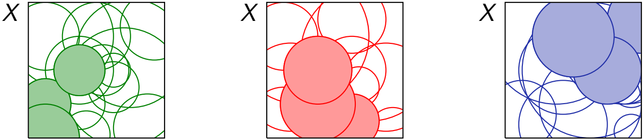

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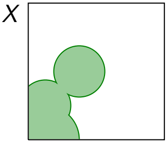

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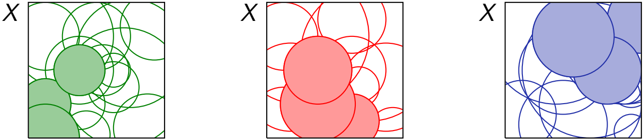

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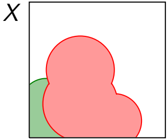
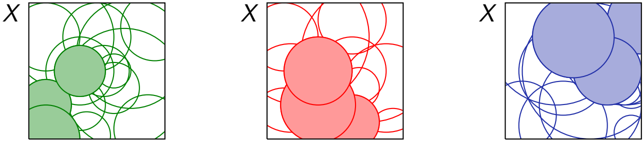


Another applications

Hurewicz's property: for every open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of X there are finite $\mathcal{V}_1 \subset \mathcal{U}_1, \mathcal{V}_2 \subset \mathcal{U}_2, \ldots$ such that $\{\, n \in \mathbb{N} : x \in \bigcup \mathcal{V}_n \,\}$ is co-finite for all $x \in X$.



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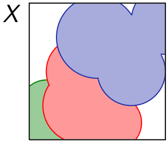

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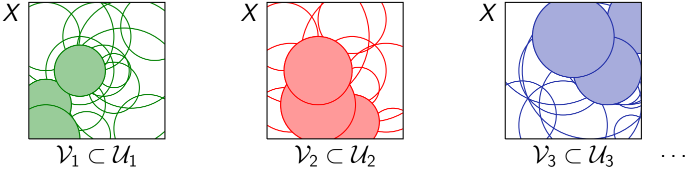
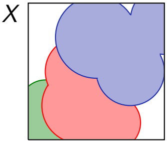


σ-compactness $\to$ Hurewicz $\to$ Menger



Another applications

Corollary ($\mathfrak{b} = \mathfrak{d}$)

For every Menger, non-Hurewicz X there is a Menger $Y \subset P(\mathbb{N})$ such that $X \times Y$ is not Menger.

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Corollary ($\mathfrak{b} = \mathfrak{d}$)

There is an ultrafilter $\mathfrak{U}$ such that in the class of sets of reals

$$\textit{Hurewicz} \not\leftrightarrow \mathfrak{U}\textit{-Menger} \not\leftrightarrow \textit{Menger}$$