The principle (\(\ast\)) of Sierpinski and a question of Miller

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The *principle (*) of Sierpinski* is the following statement:
The principle (⋆) of Sierpinski is the following statement:

There is a family of functions \( \{ \varphi_n : \omega_1 \to \omega_1 \mid n \in \omega \} \) such that for every \( I \in [\omega_1]^\omega_1 \) there is \( n \in \omega \) for which \( \varphi_n [I] = \omega_1 \).
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Arnie Miller studied this principle on his article “The onto mapping of Sierpinski” and he proved the following:
Theorem (Miller)

The following are equivalent:

1. The principle (\(\ast\)) of Sierpinski
   i.e. There is a family of functions \(\{\varphi_n : \omega_1 \rightarrow \omega_1 \mid n \in \omega\}\) such that for every \(I \in [\omega_1]^{\omega_1}\) there is \(n \in \omega\) for which \(\varphi_n[I] = \omega_1\).

2. There is a set \(X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega\) such that for every \(g : \omega \rightarrow \omega\) there is \(\alpha\) such that if \(\beta > \alpha\) then \(f_\beta \cap g\) is infinite.

The set \(X\) above resembles a Luzin set (a Luzin set is a subspace of \(\omega^\omega\) that has countable intersection with every meager set). For the purpose of this talk, we will call those sets weak Luzin sets (note that every Luzin set is a weak Luzin set).
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Thus we have the following implications:

- There is a Luzin set $\rightarrow$ There is a weak Luzin set
- $\downarrow$
- $\text{non} (\mathcal{M}) = \omega_1$
- (there is a non-meager set of size $\omega_1$)

Can this implications be reversed?
Thus we have the following implications:

There is a Luzin set \( \implies \) There is a weak Luzin set

\[ \updownarrow \]

\[ \text{non} (\mathcal{M}) = \omega_1 \]

(there is a non-meager set of size \( \omega_1 \))

Can this implications be reversed?

Miller proved that there is a weak Luzin set in the Miller model, while a theorem of Judah and Shelah says that there are no Luzin sets in such model, so the first implication can not be reversed.
This lead Miller to ask:

**Problem**

Does $\text{non} (\mathcal{M}) = \omega_1$ imply that there is a weak Luzin set?

- The answer is...
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- (an even more dramatic pause)
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**Problem**

Does $\text{non}(\mathcal{M}) = \omega_1$ imply that there is a weak Luzin set?

- The answer is...
- (Dramatic pause)
- (an even more dramatic pause)
- Yes!
We will need the following lemma:

**Lemma**

If $\text{non}(M) = \omega_1$ then there is a family $X = \{f_\alpha \mid \alpha < \omega_1\}$ with the following properties:

1. Each $f_\alpha$ is an infinite partial function from $\omega$ to $\omega$.
2. The set $\{\text{dom}(f_\alpha) \mid \alpha < \omega_1\}$ is an almost disjoint family.
3. For every $g : \omega \to \omega$ there is $\alpha < \omega_1$ such that $f_\alpha \cap g$ is infinite.
Proof.

Let $\omega^\omega = \{ s_n \mid n \in \omega \}$ and we define $H : \omega^\omega \rightarrow \text{Partial}(\omega^\omega)$ where the domain of $H(f)$ is $\{ n \mid s_n \subseteq f \}$ and if $n \in \text{dom}(H(f))$ then $H(f)(n) = f(|s_n|)$. It is easy to see that if $f \neq g$ then $\text{dom}(H(f))$ and $\text{dom}(H(g))$ are almost disjoint.

Given $g : \omega \rightarrow \omega$ we define $N(g) = \{ f \in \omega^\omega \mid |H(f) \cap g| < \omega \}$. It then follows that $N(g)$ is a meager set since $N(g) = \bigcup_{k \in \omega} N_k(g)$ where $N_k(g) = \{ f \in \omega^\omega \mid |H(f) \cap g| < k \}$ and it is easy to see that each $N_k(g)$ is a nowhere dense set. Finally, if $X = \{ h_\alpha \mid \alpha < \omega_1 \}$ is a non-meager set then $H[X]$ is the family we were looking for.

With the previous lemma we can answer Miller’s question:
**Theorem**

If \( \text{non} (\mathcal{M}) = \omega_1 \) then the principle \((\ast)\) of Sierpinski is true.

**Proof.**

Let \( X = \{ f_\alpha \mid \alpha < \omega_1 \} \) be a family as in the previous lemma. We will build a weak Luzin set \( Y = \{ h_\alpha \mid \alpha < \omega_1 \} \). For simplicity, we may assume \( \{ \text{dom} (f_n) \mid n \in \omega \} \) is a partition of \( \omega \).

For each \( n \in \omega \), let \( h_n \) be any constant function. Given \( \alpha \geq \omega \), enumerate it as \( \alpha = \{ \alpha_n \mid n \in \omega \} \) and then we recursively define \( B_0 = \text{dom} (f_{\alpha_0}) \) and \( B_{n+1} = \text{dom} (f_{\alpha_n}) \setminus (B_0 \cup \ldots \cup B_n) \). Clearly \( \{ B_n \mid n \in \omega \} \) is a partition of \( \omega \). Let \( h_\alpha = \bigcup_{n \in \omega} f_{\alpha_n} \upharpoonright B_n \), it then follows that \( Y = \{ h_\alpha \mid \alpha < \omega_1 \} \) is a weak Luzin set.
It is not hard to see that the weak Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a non-meager weak Luzin set from non \((\mathcal{M}) = \omega_1\).
It is not hard to see that the weak Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a non-meager weak Luzin set from \( \text{non}(\mathcal{M}) = \omega_1 \).

However this is not the case. This will be achieved by using Todorcevic’s method of forcing with models as side conditions.
Definition

We define the forcing $\mathbb{P}_{cat}$ as the set of all $p = (s_p, \overline{M}_p, F_p)$ with the following properties:

1. $s_p \in \omega^{<\omega}$ (this is usually referred as the stem of $p$).
2. $\overline{M}_p = \{M_0, ..., M_n\}$ is an $\in$-chain of countable elementary submodels of $H(\mathcal{(2^c)^{++}})$.
3. $F_p : \overline{M}_p \longrightarrow \omega^\omega$.
4. $s_p \cap F_p(M_i) = \emptyset$ for every $i \leq n$.
5. $F_p(M_i) \notin M_i$ and if $i < n$ then $F_p(M_i) \in M_{i+1}$.
6. $F_p(M_i)$ is a Cohen real over $M_i$ (i.e. if $Y \in M_i$ is a meager set then $F_p(M_i) \notin Y$).

Finally, if $p, q \in \mathbb{P}_{cat}$ then $p \leq q$ if $s_q \subseteq s_p$, $\overline{M}_q \subseteq \overline{M}_p$ and $F_q \subseteq F_p$. 
Theorem

The $\mathbb{P}_{cat}$ forcing has the following properties:

1. It is proper (hint: apply the “usual side conditions trick”).
2. If $X$ is a non-meager set then $\mathbb{P}_{cat}$ adds a function that has finite intersection with uncountably many elements of $X$.
3. $\mathbb{P}_{cat}$ does not destroy category (i.e. $\mathbb{P}_{cat}$ does not turn the ground model into a meager set).
4. Moreover, the iteration of the $\mathbb{P}_{cat}$ forcing does not destroy category.
Theorem

If the existence of an inaccessible cardinal is consistent, then so it is the following statement: $\text{non } (\mathcal{M}) = \omega_1$ and every weak Luzin set is meager.

Proof.

Let $\mu$ be an inaccessible cardinal, we perform a countable support iteration $\{\mathbb{P}_\alpha, Q_\alpha \mid \alpha < \mu\}$ in which $Q_\alpha$ is forced by $\mathbb{P}_\alpha$ to be the $\mathbb{P}_{\text{cat}}$ forcing. It is easy to see that if $\alpha < \mu$ then $\mathbb{P}_\alpha$ has size less than $\mu$ so it has the $\mu$-chain condition and then $\mathbb{P}_\mu$ has the $\mu$-chain condition. The result then follows by the previous results.
Thank you for your attention!