The principle $(*)$ of Sierpinski and a question of Miller

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- Sierpinski showed this is a consequence of the Continuum Hypothesis.
- Arnie Miller studied this principle on his article "The onto mapping of Sierpinski" and he proved the following:

Theorem (Miller)

The following are equivalent:

1. *The principle* $(*)$ *of Sierpinski*
   i.e. There is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ *such that for every* $I \in [\omega_1]^{\omega_1}$ *there is* $n \in \omega$ *for which* $\varphi_n[I] = \omega_1$.

2. *There is a set* $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ *such that for every* $g : \omega \longrightarrow \omega$ *there is* α *such that if* $\beta > \alpha$ *then* $f_\beta \cap g$ *is infinite.*

The set X above resembles a Luzin set (a Luzin set is a subspace of ω^ω that has countable intersection with every meager set). For the purpose of this talk, we will call those sets *weak Luzin sets* (note that every Luzin set is a weak Luzin set).

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- Thus we have the following implications:

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- Can this implications be reversed?

- Miller proved that there is a weak Luzin set in the Miller model, while a theorem of Judah and Shelah says that there are no Luzin sets in such model, so the first implication can not be reversed.

This lead Miller to ask:

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Does non $(\mathcal{M}) = \omega_1$ *imply that there is a weak Luzin set?*

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- The answer is...
- (Dramatic pause)
- (an even more dramatic pause)
- Yes!

We will need the following lemma:

Lemma

If non $(\mathcal{M}) = \omega_1$ *then there is a family* $X = \{f_\alpha \mid \alpha < \omega_1\}$ *with the following properties:*

1. *Each* f_α *is an infinite partial function from* ω *to* ω.
2. *The set* $\{dom(f_\alpha) \mid \alpha < \omega_1\}$ *is an almost disjoint family.*
3. *For every* $g : \omega \longrightarrow \omega$ *there is* $\alpha < \omega_1$ *such that* $f_\alpha \cap g$ *is infinite.*

Proof.

Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$ and we define $H : \omega^\omega \longrightarrow Partial\left(\omega^\omega\right)$ where the domain of $H(f)$ is $\{n \mid s_n \sqsubseteq f\}$ and if $n \in dom(H(f))$ then $H(f)(n) = f(|s_n|)$. It is easy to see that if $f \neq g$ then $dom(H(f))$ and $dom(H(g))$ are almost disjoint.

Given $g : \omega \longrightarrow \omega$ we define $N(g) = \{f \in \omega^\omega \mid |H(f) \cap g| < \omega\}$. It then follows that $N(g)$ is a meager set since $N(g) = \bigcup_{k \in \omega} N_k(g)$ where $N_k(g) = \{f \in \omega^\omega \mid |H(f) \cap g| < k\}$ and it is easy to see that each $N_k(g)$ is a nowhere dense set. Finally, if $X = \{h_\alpha \mid \alpha < \omega_1\}$ is a non-meager set then $H[X]$ is the family we were looking for. □

With the previous lemma we can answer Miller's question:

Theorem

If $\text{non}(\mathcal{M}) = \omega_1$ *then the principle* $(*)$ *of Sierpinski is true.*

Proof.

Let $X = \{f_\alpha \mid \alpha < \omega_1\}$ be a family as in the previous lemma. We will build a weak Luzin set $Y = \{h_\alpha \mid \alpha < \omega_1\}$. For simplicity, we may assume $\{dom(f_n) \mid n \in \omega\}$ is a partition of ω.

For each $n \in \omega$, let h_n be any constant function. Given $\alpha \geq \omega$, enumerate it as $\alpha = \{\alpha_n \mid n \in \omega\}$ and then we recursively define $B_0 = dom(f_{\alpha_0})$ and $B_{n+1} = dom(f_{\alpha_n}) \setminus (B_0 \cup ... \cup B_n)$. Clearly $\{B_n \mid n \in \omega\}$ is a partition of ω. Let $h_\alpha = \bigcup_{n \in \omega} f_{\alpha_n} \restriction B_n$, it then follows that $Y = \{h_\alpha \mid \alpha < \omega_1\}$ is a weak Luzin set. □

- It is not hard to see that the weak Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a non-meager weak Luzin set from $\text{non}(\mathcal{M}) = \omega_1$.

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- However this is not the case. This will be achieved by using Todorcevic's method of forcing with models as side conditions.

Definition

We define the forcing $\mathbb{P}_{cat}$ as the set of all $p = \left(s_p, \overline{M}_p, F_p\right)$ with the following properties:

1. $s_p \in \omega^{<\omega}$ (this is usually referred as *the stem* of p).
2. $\overline{M}_p = \{M_0, ..., M_n\}$ is an $\in$-chain of countable elementary submodels of $\mathrm{H}(\left(2^{\mathfrak{c}}\right)^{++})$.
3. $F_p : \overline{M}_p \longrightarrow \omega^{\omega}$.
4. $s_p \cap F_p\left(M_i\right) = \emptyset$ for every $i \leq n$.
5. $F_p\left(M_i\right) \notin M_i$ and if $i < n$ then $F_p\left(M_i\right) \in M_{i+1}$.
6. $F_p\left(M_i\right)$ is a Cohen real over M_i (i.e. if $Y \in M_i$ is a meager set then $F_p\left(M_i\right) \notin Y$).

Finally, if $p, q \in \mathbb{P}_{cat}$ then $p \leq q$ if $s_q \subseteq s_p$, $\overline{M}_q \subseteq \overline{M}_p$ and $F_q \subseteq F_p$.

Theorem

The $\mathbb{P}_{cat}$ forcing has the following properties:

1. *It is proper (hint: apply the "usual side conditions trick").*
2. *If X is a non-meager set then $\mathbb{P}_{cat}$ adds a function that has finite intersection with uncountably many elements of X.*
3. *$\mathbb{P}_{cat}$ does not destroy category (i.e. $\mathbb{P}_{cat}$ does not turn the ground model into a meager set).*
4. *Moreover, the iteration of the $\mathbb{P}_{cat}$ forcing does not destroy category.*

Theorem

If the existence of an inaccessible cardinal is consistent, then so it is the following statement: $\mathrm{non}(\mathcal{M}) = \omega_1$ *and every weak Luzin set is meager.*

Proof.

Let μ be an inaccessible cardinal, we perform a countable support iteration $\{\mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \mu\}$ in which $\mathbb{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be the $\mathbb{P}_{cat}$ forcing. It is easy to see that if $\alpha < \mu$ then $\mathbb{P}_\alpha$ has size less than μ so it has the μ-chain condition and then $\mathbb{P}_\mu$ has the μ-chain condition. The result then follows by the previous results. □

Thank You!