(Non)measurability of $\mathcal{I}$-Luzin sets

joint work with Szymon Żeberski

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We live in the Euclidean space $\mathbb{R}^n$ and work with ZFC.

Definition

For each $A, B \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ *we define:*

$$\begin{aligned} A + B &= \{a + b : \ a \in A, b \in B\}, \\ x + A &= \{x\} + A, \end{aligned}$$

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Let's denote a family of Borel sets by $\mathcal{B}$.

Definition

We say that a σ*-ideal* $\mathcal{I}$*:*

- *is translation invariant if for each* $x \in \mathbb{R}^n$ *and* $A \in \mathcal{I}$ *we have* $x + A \in \mathcal{I}$*;*
- *has a Borel base if* $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B} \cap \mathcal{I})(I \subseteq B)$

We shall consider proper, containing all countable sets σ-ideals with a Borel base only.

Definition

We say that a set A is:

- $\mathcal{I}$*-residual if A is a complement of some set $I \in \mathcal{I}$;*
- $\mathcal{I}$*-positive Borel set if $A \in \mathcal{B} \setminus \mathcal{I}$;*
- $\mathcal{I}$*-nonmeasurable if A doesn't belong to the σ-field $\sigma(\mathcal{B} \cup \mathcal{I})$ generated by Borel sets and the σ-ideal $\mathcal{I}$;*
- *completely* $\mathcal{I}$*-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $\mathcal{I}$-positive Borel set B.*

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Example

Bernstein sets are completely $\mathcal{I}$-nonmeasurable with respect to any reasonable $\mathcal{I}$.

Definition

We say that a set A is an $\mathcal{I}$-Luzin set, if for each $I \in \mathcal{I}$ we have $|A \cap I| < |A|$.
A is called a super $\mathcal{I}$-Luzin set, if A is an $\mathcal{I}$-Luzin set and for each $\mathcal{I}$-positive Borel set B we have $|A \cap B| = |A|$.

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Example

For $\mathcal{M}$ and $\mathcal{N}$ σ-ideals of meager and null sets respectively we call a $\mathcal{M}$-Luzin set a generalized Luzin set and a $\mathcal{N}$-Luzin set a generalized Sierpiński set.

Definition

$\mathcal{I}$ has a Weaker Smital Property, if there exists a countable dense set D such that for each $\mathcal{I}$-positive Borel set A a set $A + D$ is $\mathcal{I}$-residual.
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$\mathcal{I}$ has a Smital Property if $A + D$ is $\mathcal{I}$-residual for each $\mathcal{I}$-positive Borel set A and each dense set D.
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Proposition

Steinhaus Property $\Rightarrow$ Smital Property $\Rightarrow$ Weaker Smital Property.

Example

Classic examples of σ-ideals that have all of the stated properties are $\mathcal{M}$ and $\mathcal{N}$. On the other hand a σ-ideal of meager null sets $\mathcal{M} \cap \mathcal{N}$ doesn't.

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Theorem

Let $\mathcal{I}$ be a translation invariant σ-ideal possesing the Weaker Smital Property. Then every $\mathcal{I}$-Luzin set is $\mathcal{I} -$ nonmeasurable.

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$\mathcal{I}$-Luzin sets are $\mathcal{I} -$ nonmeasurable $\Leftrightarrow$ Every $\mathcal{I}$-positive Borel set B contains a perfect subset from $\mathcal{I}$.

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- Borel base: take $I \subseteq I' \in \mathcal{B} \cap \mathcal{I}$, then $B\backslash I' \subseteq X$;
- $B\backslash I'$ is $\mathcal{I}$-positive, so it contains some perfect set from $\mathcal{I}$ against the assumption that X is an $\mathcal{I}$-Luzin set.

Proof ctnd.

$\Rightarrow$: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set B without the mentioned property. We claim that B is itself an $\mathcal{I}$-Luzin set.

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- What means that B is a Borel $\mathcal{I}$-Luzin set.

Lemma

If $\mathcal{I}$-Luzin set exists then there exists an $\mathcal{I}$-Luzin of regular cardinality.

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Theorem

Let's assume that σ-ideal $\mathcal{I}$ has the Weaker Smital Property. Then if A is an $\mathcal{I}$-Luzin set of regular cardinality then $D + A$ is a super $\mathcal{I}$-Luzin set (D witnesses the Weaker Smital Property).

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Example

$\mathcal{N}$ and $\mathcal{M}$ have the Weaker Smital Property. $\mathcal{N} \cap \mathcal{M}$ doesn't have the Weaker Smital Property but still $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable. For σ-ideal of countable sets $[\mathbb{R}]^{\leq\omega}$ whole space $\mathbb{R}^n$ is an $\mathcal{I}$-Luzin set.

Question

What conditions should a σ-ideal $\mathcal{I}$ meet to allow transformation of $\mathcal{I}$-Luzin sets into super $\mathcal{I}$-Luzin sets?

Theorem

Assume that $add(\mathcal{I}) = \mathfrak{c}$. *Then there exists an* $\mathcal{I}$-*Luzin set* X *such that* $X + X$ *is a Bernstein set.*

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Under right assumptions there exists a generalized Luzin (Sierpiński) set X *such that* $X + X$ *is a Bernstein set.*

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Under right assumptions there exists a generalized Luzin (Sierpiński) set X *such that* $X + X$ *is a Bernstein set.*

What about $L + S$, where L is a Luzin set and S is a Sierpiński set?

Theorem (Babinkostova, Scheepers, 2007)

Let L be a classic Luzin set and S be a classic Sierpiński. Then $L + S$ is not a Bernstein set since it's Menger.

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Assume that $\mathfrak{c}$ is a regular cardinal. Then $L + S$, where L is a generalized Luzin set and S is a generalized Sierpiński set, belongs to Marczewski ideal s_0.

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Definition

Recall that a set $A \in s_0$ if

$$(\forall P\text{-perfect})\ (\exists Q\text{-perfect})\ (Q \subseteq P \text{ and } Q \cap A = \emptyset)$$

Lemma

For every compact null set P there exists a comeager set G such that $G + P$ is still null.

Proof of the Theorem.

If $|L+S|<\mathfrak{c}$ then there is nothing to prove. Otherwise $|L|=|S|=\mathfrak{c}$ by regularity of $\mathfrak{c}$. Let P be an arbitrary chosen perfect set P (wlog- meager, null and compact) and let G be as in the previous Lemma. Let's denote $N=-G$ and $M=-N^c$. Then $P\subseteq(M+N)^c$. We will show that also $(L+S)^c$ also contains some perfect set.

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$$\begin{aligned} L+S &= ((L\cap N)+(S\cap M))\cup((L\cap N)+(S\cap M^c))\cup \\ &\cup((L\cap N^c)+(S\cap M))\cup((L\cap N^c)+(S\cap M^c)) \end{aligned}$$

- $(L\cap N)+(S\cap M)\subseteq M+N$;
- $(L\cap N)+(S\cap M^c)$ is a Luzin set;
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- $|(L\cap N^c)+(S\cap M^c)|<\mathfrak{c}$.

It follows that all of these sets have intersection with P of power lesser than $\mathfrak{c}$, so there exists perfect set $P'\subseteq P$ such that $P'\subseteq(L+S)^c$. Thus $L+S$ belongs to s_0.

Remark

The assumption on regularity of $\mathfrak{c}$ cannot be omitted.

Thank you for your attention!

Bibliography

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