(Non)measurability of $\mathcal{I}$-Luzin sets

joint work with Szymon Żeberski

Marcin Michalski

Wrocław University of Technology

Winter School in Abstract Analysis 2016, section Set Theory and Topology
30.01 - 06.02.2016, Hejnice
We live in the Euclidean space $\mathbb{R}^n$ and work with ZFC.

**Definition**

For each $A, B \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ we define:

\[
A + B = \{a + b : a \in A, b \in B\},
\]

\[
x + A = \{x\} + A,
\]
We live in the Euclidean space $\mathbb{R}^n$ and work with ZFC.

**Definition**

For each $A, B \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ we define:

\[ A + B = \{ a + b : a \in A, b \in B \}, \]
\[ x + A = \{ x \} + A, \]

Let’s denote a family of Borel sets by $\mathcal{B}$.

**Definition**

We say that a $\sigma$-ideal $\mathcal{I}$:

- is translation invariant if for each $x \in \mathbb{R}^n$ and $A \in \mathcal{I}$ we have $x + A \in \mathcal{I}$;
- has a Borel base if $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B} \cap \mathcal{I})(I \subseteq B)$

We shall consider proper, containing all countable sets $\sigma$-ideals with a Borel base only.
We say that a set $A$ is:

- $\mathcal{I}$-residual if $A$ is a complement of some set $I \in \mathcal{I}$;
- $\mathcal{I}$-positive Borel set if $A \in \mathcal{B}\setminus \mathcal{I}$;
- $\mathcal{I}$-nonmeasurable if $A$ doesn't belong to the $\sigma$-field $\sigma(\mathcal{B} \cup \mathcal{I})$ generated by Borel sets and the $\sigma$-ideal $\mathcal{I}$;
- completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $\mathcal{I}$-positive Borel set $B$. 

Example

Bernstein sets are completely $\mathcal{I}$-nonmeasurable with respect to any reasonable $\mathcal{I}$. 

Marcin Michalski | (Non)measurability of $\mathcal{I}$-Luzin sets
**Definition**

We say that a set $A$ is:

- $\mathcal{I}$-residual if $A$ is a complement of some set $I \in \mathcal{I}$;
- $\mathcal{I}$-positive Borel set if $A \in \mathcal{B}\setminus\mathcal{I}$;
- $\mathcal{I}$-nonmeasurable if $A$ doesn’t belong to the $\sigma$-field $\sigma(\mathcal{B} \cup \mathcal{I})$ generated by Borel sets and the $\sigma$-ideal $\mathcal{I}$;
- completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $\mathcal{I}$-positive Borel set $B$.

**Example**

Bernstein sets are completely $\mathcal{I}$-nonmeasurable with respect to any reasonable $\mathcal{I}$. 
Definition

We say that a set $A$ is an $\mathcal{I}$-Luzin set, if for each $I \in \mathcal{I}$ we have $|A \cap I| < |A|$. 

$A$ is called a super $\mathcal{I}$-Luzin set, if $A$ is an $\mathcal{I}$-Luzin set and for each $\mathcal{I}$-positive Borel set $B$ we have $|A \cap B| = |A|$. 

Example

For $\mathcal{M}$ and $\mathcal{N}$ $\sigma$-ideals of meager and null sets respectively we call a $\mathcal{M}$-Luzin set a generalized Luzin set and a $\mathcal{N}$-Luzin set a generalized Sierpiński set.
**Definition**

We say that a set $A$ is an $\mathcal{I}$-Luzin set, if for each $I \in \mathcal{I}$ we have $|A \cap I| < |A|$.

A is called a super $\mathcal{I}$-Luzin set, if $A$ is an $\mathcal{I}$-Luzin set and for each $\mathcal{I}$-positive Borel set $B$ we have $|A \cap B| = |A|$.

**Example**

For $\mathcal{M}$ and $\mathcal{N}$ $\sigma$-ideals of meager and null sets respectively we call a $\mathcal{M}$-Luzin set a generalized Luzin set and a $\mathcal{N}$-Luzin set a generalized Sierpiński set.
Definition

\( \mathcal{I} \) has a Weaker Smital Property, if there exists a countable dense set \( D \) such that for each \( \mathcal{I} \)-positive Borel set \( A \) a set \( A + D \) is \( \mathcal{I} \)-residual. We say that the set \( D \) witnesses that \( \mathcal{I} \) has the Weaker Smital Property.

Definition

\( \mathcal{I} \) has a Smital Property if \( A + D \) is \( \mathcal{I} \)-residual for each \( \mathcal{I} \)-positive Borel set \( A \) and each dense set \( D \).

\( \mathcal{I} \) has a Steinhaus Property if for every \( \mathcal{I} \)-positive Borel sets \( A \) and \( B \) a set \( A + B \) has nonempty interior.

Proposition

Steinhaus Property \( \Rightarrow \) Smital Property \( \Rightarrow \) Weaker Smital Property.

Marcin Michalski | (Non)measurability of \( \mathcal{I} \)-Luzin sets
**Definition**

$I$ has a Weaker Smital Property, if there exists a countable dense set $D$ such that for each $I$-positive Borel set $A$ a set $A + D$ is $I$-residual. We say that the set $D$ witnesses that $I$ has the Weaker Smital Property.

**Definition**

$I$ has a Smital Property if $A + D$ is $I$-residual for each $I$-positive Borel set $A$ and each dense set $D$.

$I$ has a Steinhaus Property if for every $I$-positive Borel sets $A$ and $B$ a set $A + B$ has nonempty interior.
Definition

$I$ has a Weaker Smital Property, if there exists a countable dense set $D$ such that for each $I$-positive Borel set $A$ a set $A + D$ is $I$-residual. We say that the set $D$ witnesses that $I$ has the Weaker Smital Property.

Definition

$I$ has a Smital Property if $A + D$ is $I$-residual for each $I$-positive Borel set $A$ and each dense set $D$.

$I$ has a Steinhaus Property if for every $I$-positive Borel sets $A$ and $B$ a set $A + B$ has nonempty interior.

Proposition

Steinhaus Property $\Rightarrow$ Smital Property $\Rightarrow$ Weaker Smital Property.
Example

Classic examples of $\sigma$-ideals that have all of the stated properties are $\mathcal{M}$ and $\mathcal{N}$. On the other hand a $\sigma$-ideal of meager null sets $\mathcal{M} \cap \mathcal{N}$ doesn’t.
Example

Classic examples of $\sigma$-ideals that have all of the stated properties are $\mathcal{M}$ and $\mathcal{N}$. On the other hand a $\sigma$-ideal of meager null sets $\mathcal{M} \cap \mathcal{N}$ doesn’t.

Theorem

Let $\mathcal{I}$ be a translation invariant $\sigma$-ideal possessing the Weaker Smital Property. Then every $\mathcal{I}$-Luzin set is $\mathcal{I}$–nonmeasurable.
Theorem

$I$-Luzin sets are $I$ – nonmeasurable ⇔ Every $I$-positive Borel set $B$ contains a perfect subset from $I$.

Proof. $\Leftarrow$: By contradiction. Suppose that we have an $I$-measurable $I$-Luzin set $X$. Then $X = B \Delta I$, $B \in B \setminus I$, $I \in I$; Borel base: take $I \subseteq I' \in B \cap I$, then $B \setminus I' \subseteq X$; $B \setminus I'$ is $I$-positive, so it contains some perfect set from $I$ against the assumption that $X$ is an $I$-Luzin set.
**Theorem**

$I$-Luzin sets are $I$– nonmeasurable $\iff$ Every $I$-positive Borel set $B$ contains a perfect subset from $I$.

**Proof.**

$\Leftarrow$: By contradiction. Suppose that we have an $I$-measurable $I$-Luzin set $X$.

$\Rightarrow$: [Further proof steps]

Marcin Michalski  |  (Non)measurability of $I$-Luzin sets
Theorem

$I$-Luzin sets are $I$– nonmeasurable $\iff$ Every $I$-positive Borel set $B$ contains a perfect subset from $I$.

Proof.

$\Leftarrow$: By contradiction. Suppose that we have an $I$-measurable $I$-Luzin set $X$.

- $X = B \triangle l$, $B \in B \setminus I$, $l \in I$;
Theorem

$I$-Luzin sets are $I$ – nonmeasurable $\iff$ Every $I$-positive Borel set $B$ contains a perfect subset from $I$.

Proof.

$\Leftarrow$: By contradiction. Suppose that we have an $I$-measurable $I$-Luzin set $X$.

- $X = B \Delta I, B \in B \setminus I, I \in I$;

- Borel base: take $I \subseteq I' \in B \cap I$, then $B \setminus I' \subseteq X$;
Theorem

$I$-Luzin sets are $I$ – nonmeasurable $\iff$ Every $I$-positive Borel set $B$ contains a perfect subset from $I$.

Proof.

$\Leftarrow$: By contradiction. Suppose that we have an $I$-measurable $I$-Luzin set $X$.

- $X = B \Delta I$, $B \in B \setminus I$, $I \in I$;
- Borel base: take $I \subseteq I' \in B \cap I$, then $B \setminus I' \subseteq X$;
- $B \setminus I'$ is $I$-positive, so it contains some perfect set from $I$ against the assumption that $X$ is an $I$-Luzin set.
Proof ctnd.

⇒: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set $B$ without the mentioned property. We claim that $B$ is itself an $\mathcal{I}$-Luzin set.
Proof ctnd.

⇒: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set $B$ without the mentioned property. We claim that $B$ is itself an $\mathcal{I}$-Luzin set.

- If it’s not, then there is $I \in \mathcal{I}$ for which $|B \cap I| = c$;
Proof ctnd.

⇒: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set $B$ without the mentioned property. We claim that $B$ is itself an $\mathcal{I}$-Luzin set.

- If it’s not, then there is $I \in \mathcal{I}$ for which $|B \cap I| = c$;
- Borel base: we may assume that $I$ is Borel and thus $B \cap I$ is a Borel set from $\mathcal{I}$;

By the Perfect Set Property $B \cap I$ (and so $B$ alone) contains some perfect set $P \in \mathcal{I}$, against the assumptions; what means that $B$ is a Borel $\mathcal{I}$-Luzin set.
Proof ctnd.

⇒: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set $B$ without the mentioned property. We claim that $B$ is itself an $\mathcal{I}$-Luzin set.

- If it’s not, then there is $I \in \mathcal{I}$ for which $|B \cap I| = c$;
- Borel base: we may assume that $I$ is Borel and thus $B \cap I$ is a Borel set from $\mathcal{I}$;
- By the Perfect Set Property $B \cap I$ (and so $B$ alone) contains some perfect set $P \in \mathcal{I}$, against the assumptions;
Proof ctnd.

⇒: Also by contradiction. Suppose that we have a Borel $\mathcal{I}$-positive set $B$ without the mentioned property. We claim that $B$ is itself an $\mathcal{I}$-Luzin set.

- If it’s not, then there is $I \in \mathcal{I}$ for which $|B \cap I| = c$;
- Borel base: we may assume that $I$ is Borel and thus $B \cap I$ is a Borel set from $\mathcal{I}$;
- By the Perfect Set Property $B \cap I$ (and so $B$ alone) contains some perfect set $P \in \mathcal{I}$, against the assumptions;
- What means that $B$ is a Borel $\mathcal{I}$-Luzin set.
Lemma

If $\mathcal{I}$-Luzin set exists then there exists an $\mathcal{I}$-Luzin of regular cardinality.
Lemma

If $\mathcal{I}$-Luzin set exists then there exists an $\mathcal{I}$-Luzin of regular cardinality.

Theorem

Let’s assume that $\sigma$-ideal $\mathcal{I}$ has the Weaker Smital Property. Then if $A$ is an $\mathcal{I}$-Luzin set of regular cardinality then $D + A$ is a super $\mathcal{I}$-Luzin set ($D$ witnesses the Weaker Smital Property).
Lemma

If $\mathcal{I}$-Luzin set exists then there exists an $\mathcal{I}$-Luzin of regular cardinality.

Theorem

Let’s assume that $\sigma$-ideal $\mathcal{I}$ has the Weaker Smital Property. Then if $A$ is an $\mathcal{I}$-Luzin set of regular cardinality then $D + A$ is a super $\mathcal{I}$-Luzin set ($D$ witnesses the Weaker Smital Property).

Example

$\mathcal{N}$ and $\mathcal{M}$ have the Weaker Smital Property. $\mathcal{N} \cap \mathcal{M}$ doesn’t have the Weaker Smital Property but still $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable. For $\sigma$-ideal of countable sets $[\mathbb{R}]^{\leq \omega}$ whole space $\mathbb{R}^n$ is an $\mathcal{I}$-Luzin set.
Question

What conditions should a $\sigma$-ideal $\mathcal{I}$ meet to allow transformation of $\mathcal{I}$-Luzin sets into super $\mathcal{I}$-Luzin sets?
Theorem

Assume that $\text{add}(\mathcal{I}) = \mathfrak{c}$. Then there exists an $\mathcal{I}$-Luzin set $X$ such that $X + X$ is a Bernstein set.

Corollary

Under right assumptions there exists a generalized Luzin (Sierpiński) set $X$ such that $X + X$ is a Bernstein set.

What about $L + S$, where $L$ is a Luzin set and $S$ is a Sierpiński set?
Theorem

Assume that $\text{add}(\mathcal{I}) = c$. Then there exists an $\mathcal{I}$-Luzin set $X$ such that $X + X$ is a Bernstein set.

Corollary

Under right assumptions there exists a generalized Luzin (Sierpiński) set $X$ such that $X + X$ is a Bernstein set.
Theorem

Assume that \( \text{add}(\mathcal{I}) = \mathfrak{c} \). Then there exists an \( \mathcal{I} \)-Luzin set \( X \) such that \( X + X \) is a Bernstein set.

Corollary

Under right assumptions there exists a generalized Luzin (Sierpiński) set \( X \) such that \( X + X \) is a Bernstein set.

What about \( L + S \), where \( L \) is a Luzin set and \( S \) is a Sierpiński set?
Theorem (Babinkostova, Scheepers, 2007)

Let $L$ be a classic Luzin set and $S$ be a classic Sierpiński. Then $L + S$ is not a Bernstein set since it’s Menger.
Theorem (Babinkostova, Scheepers, 2007)

Let $L$ be a classic Luzin set and $S$ be a classic Sierpiński. Then $L + S$ is not a Bernstein set since it’s Menger.

Theorem (M.M., Szymon Žeberski)

Assume that $\mathfrak{c}$ is a regular cardinal. Then $L + S$, where $L$ is a generalized Luzin set and $S$ is a generalized Sierpiński set, belongs to Marczewski ideal $s_0$. 

Definition

Recall that a set $A \in s_0$ if $(\forall P$-perfect $)(\exists Q$-perfect $)(Q \subseteq P$ and $Q \cap A = \emptyset)$.
Theorem (Babinkostova, Scheepers, 2007)

Let $L$ be a classic Luzin set and $S$ be a classic Sierpiński. Then $L + S$ is not a Bernstein set since it’s Menger.

Theorem (M.M., Szymon Žeberski)

Assume that $\mathfrak{c}$ is a regular cardinal. Then $L + S$, where $L$ is a generalized Luzin set and $S$ is a generalized Sierpiński set, belongs to Marczewski ideal $s_0$.

Definition

Recall that a set $A \in s_0$ if

$$(\forall P\text{-perfect}) \ (\exists Q\text{-perfect}) \ (Q \subseteq P \text{ and } Q \cap A = \emptyset)$$
Lemma

For every compact null set $P$ there exists a comeager set $G$ such that $G + P$ is still null.
Proof of the Theorem.

If $|L + S| < c$ then there is nothing to prove. Otherwise $|L| = |S| = c$ by regularity of $c$. Let $P$ be an arbitrary chosen perfect set $P$ (wlog- meager, null and compact) and let $G$ be as in the previous Lemma. Let’s denote $N = -G$ and $M = -N^c$. Then $P \subseteq (M + N)^c$. We will show that also $(L + S)^c$ also contains some perfect set.
Proof of the Theorem.

If $|L + S| < \mathfrak{c}$ then there is nothing to prove. Otherwise $|L| = |S| = \mathfrak{c}$ by regularity of $\mathfrak{c}$. Let $P$ be an arbitrary chosen perfect set $P$ (wlog- meager, null and compact) and let $G$ be as in the previous Lemma. Let’s denote $N = -G$ and $M = -N^c$. Then $P \subseteq (M + N)^c$. We will show that also $(L + S)^c$ also contains some perfect set.

$$L + S = ((L \cap N) + (S \cap M)) \cup ((L \cap N) + (S \cap M^c)) \cup \cup((L \cap N^c) + (S \cap M)) \cup ((L \cap N^c) + (S \cap M^c))$$

- $(L \cap N) + (S \cap M) \subseteq M + N$;
- $(L \cap N) + (S \cap M^c)$ is a Luzin set;
- $(L \cap N^c) + (S \cap M)$ is a Sierpiński set;
- $|(L \cap N^c) + (S \cap M^c)| < \mathfrak{c}$.
Proof of the Theorem.

If $|L + S| < c$ then there is nothing to prove. Otherwise $|L| = |S| = c$ by regularity of $c$. Let $P$ be an arbitrary chosen perfect set $P$ (wlog- meager, null and compact) and let $G$ be as in the previous Lemma. Let’s denote $N = -G$ and $M = -N^c$. Then $P \subseteq (M + N)^c$. We will show that also $(L + S)^c$ also contains some perfect set.

$$L + S = ((L \cap N) + (S \cap M)) \cup ((L \cap N) + (S \cap M^c)) \cup \cup((L \cap N^c) + (S \cap M)) \cup ((L \cap N^c) + (S \cap M^c))$$

- $(L \cap N) + (S \cap M) \subseteq M + N$;
- $(L \cap N) + (S \cap M^c)$ is a Luzin set;
- $(L \cap N^c) + (S \cap M)$ is a Sierpiński set;
- $|(L \cap N^c) + (S \cap M^c)| < c$.

It follows that all of these sets have intersection with $P$ of power lesser than $c$, so there exists perfect set $P' \subseteq P$ such that $P' \subseteq (L + S)^c$. Thus $L + S$ belongs to $s_0$. 

Marcin Michalski
(Non)measurability of $I$-Luzin sets
Remark

The assumption on regularity of \( c \) cannot be omitted.
Thank you for your attention!
Bibliography