Rosenthal compacta that are premetric of finite degree

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Winter School in Abstract Analysis
Hejnice, February 2016

Rosenthal compacta: Origins

Rosenthal compacta history

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Banach spaces' history

Origins: preliminary definitions

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- $(X, \|\cdot\|)$ normed

Origins: preliminary definitions

- $(X, \|\cdot\|)$ Banach

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- $(X, \|\cdot\|)$ Banach $\longrightarrow (X, \mathscr{T}_{\|\cdot\|})$ complete

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Dual and Bidual spaces of X

$$X^* = \{x^* : X \longrightarrow \mathbb{R} : \ x^* \text{ linear and } \|\cdot\|\text{-continuous}\}$$

$$X^{**} = \{x^{**} : X^* \longrightarrow \mathbb{R} : \ x^* \text{ linear and } \|\cdot\|^*\text{-continuous}\}$$

Origins: preliminary definitions

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$$B_{X^*} = \{x^* \in X^* : \|x^*\|^* \leq 1\}$$
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Origins: separable Banach spaces with separable dual

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If X is a separable Banach space, is X^* separable too? NO

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If X is a separable Banach space, is X^* separable too? ℓ^1 is separable but ℓ_∞ no

Origins: separable Banach spaces with separable dual

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If X is a separable Banach space that does not contain copies of ℓ^1 then is X^* separable too? NO $\to$ James [Jam74] and Lindestrauss & Stegall [LS75]

Orígins: Some topological definitions

Definition (Completely metrizable space)

A topological space $(X, \mathscr{T})$ is completely metrizable if

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Definition (Polish space)

A topological space $(X, \mathscr{T})$ is Polish if it is separable and completely metrizable.

Orígins: Otras definiciones

Given a Polish space X

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Given a Polish space X

Definition (Baire first one class $\mathcal{B}_1(X)$)

A function $f : X \longrightarrow \mathbb{R}$ is of the *of the Baire first one class or simply* $1-$*Baire* ($f \in \mathcal{B}_1(X)$) if it is point-wise limit of continuous functions on X.

Definition (Rosenthal compactum) [God80]

Given a Polish space X a compact space $K \subset \mathbb{R}^X$ is a Rosenthal compactum if it is homeomorphic to a point-wise subspace of $\boldsymbol{\mathcal{B}_1}(X)$.

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$$\Downarrow$$

Rosenthal compacta are *descriptible objects*.

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A characterization:

Theorem (Odell & Rosenthal) [OR75]

Given a separable Banach space X then the following statements are equivalent

- X does not contain copies of ℓ^1.
- $B_{X^{**}}$ is a separable Rosenthal compactum in $\boldsymbol{\mathcal{B}}_{\mathbf{1}}(B_{X^*})$.

Rosenthal compacta: definition, examples and properties

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- $B_{X^{**}}$ when X is a separable Banach space without copies of ℓ^1.

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- The Split Interval $S(I)$ and the Alexandroff Duplicate $D(2^{\mathbb{N}})$.

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- The Split Interval $S(I)$ and the Alexandroff Duplicate $D(2^{\mathbb{N}})$.
- The $n-$Split Interval $S_n(I)$ and the Alexandroff $n-$plicate $D_n(2^{\mathbb{N}})$.

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- A Rosenthal compactum K is hereditarily separable iff it does not contain discrete subspaces of uncontable size. ([Pol84][Tod99])

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- Every Rosenthal compacta have cardinality at most $\mathfrak{c}$.
- Rosenthal compacta are Fréchet-Uryshon spaces ([BFT78][Ros77])
- A Rosenthal compactum K is hereditarily separable iff it does not contain discrete subspaces of uncontable size. ([Pol84] [Tod99])
- Given any separable compactum K ([God80])

$$K \text{ Rosenthal } \iff \forall E \hookrightarrow \ell_\infty \text{ si } E \cong \mathscr{C}(K) \Rightarrow E \in \mathbf{\Sigma^1_1}(\mathbb{R}^{\mathbb{N}})$$

The Split Interval $S(I)$.

Let $I = [0, 1]$,

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We define the **Split Interval** $S(I)$, as the space $I \times \{0, 1\} \setminus \{(0, 0), (1, 1)\}$ endowed with the topology induced by the lexicographical order.

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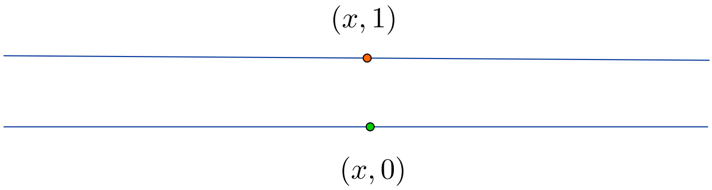

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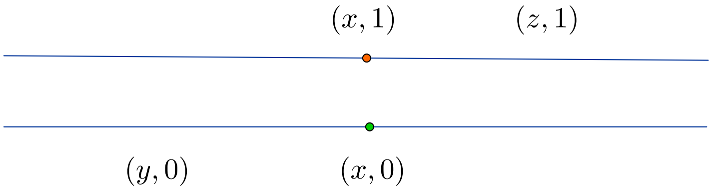

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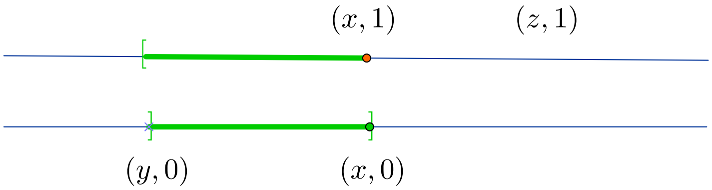

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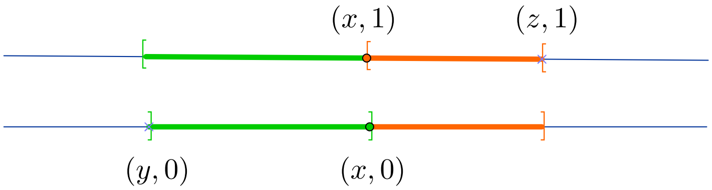

Some properties of $S(I)$.

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- $S(I)$ is a Rosenthal compactum.

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- $S(I)$ is hereditarily separable.
- The unique metrizable subspaces of $S(I)$ are the countable ones. In particular, $S(I)$ is not metrizable.

The Alexandroff duplicate $D(2^{\mathbb{N}})$.

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Definition (Alexandroff Duplicate)

Given any topological Hausdorff space $(X, \mathscr{T})$ we define the Alexandroff Duplicate $D(X)$ as the space $X \times \{0,1\}$ endowed with the topology for which the points $(x,1)$ are isolated and the points $(x,0)$ have the following neighbourhoods

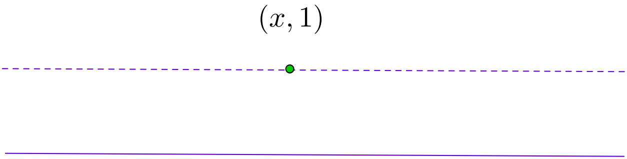
$$\mathscr{U} \times \{0,1\} \setminus \{(x,1)\}$$

where $\mathscr{U}$ is an open neighbourhood of x in X.

We are only interested in $X = 2^{\mathbb{N}}$.

Alexandroff Duplicate $D(2^{\mathbb{N}})$.

$2^{\mathbb{N}} \times \{0, 1\}$



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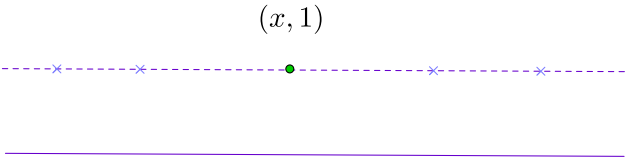

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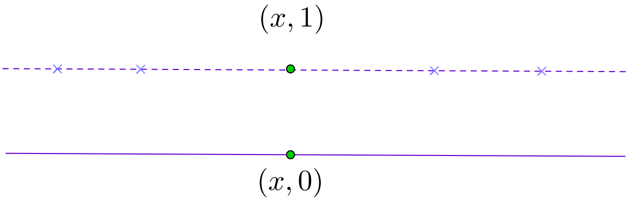

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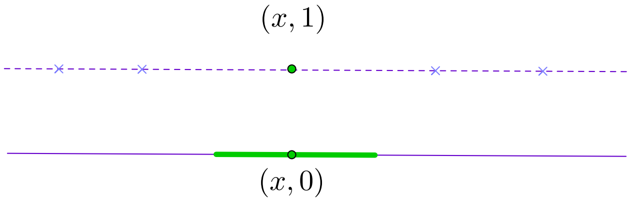

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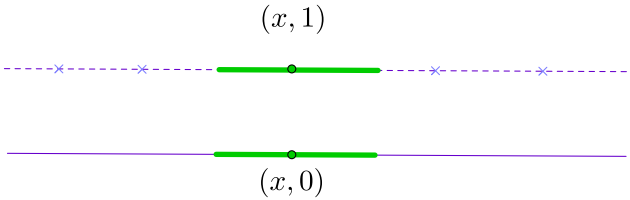

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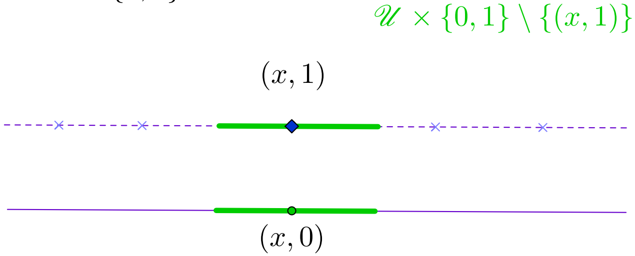

Some properties of $D(2^{\mathbb{N}})$.

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- $D(2^{\mathbb{N}})$ is a Rosenthal compactum [Tod99]. (Non-separable)
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- $D(2^{\mathbb{N}})$ is a Rosenthal compactum [Tod99]. (Non-separable)
- $D(2^{\mathbb{N}})$ is not metrizable.
- $D(2^{\mathbb{N}})$ is monolithic. (i.e. every separable subspace is metrizable)

The compact spaces $S(I)$ and $D(2^{\mathbb{N}})$ and the structure theorems.

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Definition (Pre-metric compactum of degree at most n)

Given any natural number $n \geq 1$ we say that a Rosenthal compactum K is a pre-metric compactum of degree at most n if

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Definition (Pre-metric compactum of degree at most n)

Given any natural number $n \geq 1$ we say that a Rosenthal compactum K is a pre-metric compactum of degree at most n if there exists a continuous surjective function f from K onto a compact metric space M which is at most n to 1:

$$|f^{-1}(x)| \leq n, \ \ \forall x \in M$$

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Theorem (S. Todorcevic [Tod99])

Let K a Rosenthal compactum that is not metric itself then

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Theorem (S. Todorcevic [Tod99])

Let K a Rosenthal compactum that is not metric itself then either contains an uncountable discrete subspace or it is a pre-metric compactum of degree at most 2.

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Theorem (S. Todorcevic [Tod99])

Let K a Rosenthal compactum that is not metric itself then either contains an uncountable discrete subspace or it contains copies of $S(I)$.

The compact spaces $S(I)$ and $D(2^{\mathbb{N}})$ and the structure theorems.

Is there some way to
classify the structure of pre-metric compacta
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The compact spaces $S(I)$ and $D(2^{\mathbb{N}})$ and the structure theorems.

Theorem (S. Todorcevic) [Tod99]

For a separable Rosenthal compactum K which is pre-metric of degree at most 2 one of the following alternatives holds:

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Theorem (S. Todorcevic) [Tod99]

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3. K contains copies of $D(2^{\mathbb{N}})$.

Two basic facts

1. K is metric if and only if it is a pre-metric compactum of degree at most 1.

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1. K is metric if and only if it is a pre-metric compactum of degree at most 1.
2. $S(I)$ and $D(2^{\mathbb{N}})$ are pre-metric compacta of degree at most 2 but not of degree at most 1.

Restating Todorcevic's result

Theorem (S. Todorcevic) [Tod99]

A separable Rosenthal compactum K that is pre-metric of degree at most 2 satisfies one of the following alternatives:

1. K is a pre-metric compactum of degree at most 1.
2. K contains copies of $S(I)$.
3. K contains copies of $D(2^{\mathbb{N}})$.

A more general question

Is there a similar classification for separable Rosenthal compacta which are pre-metric of degree at most n ($n \geq 3$)?

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Is there a similar classification for separable Rosenthal compacta which are pre-metric of degree at most n $(n \geq 3)$? YES

Our work: the main result

Our main result is:

Theorem (Avilés, P. , Todorcevic)[APT]

Fix a natural number $n \geq 2$. If a separable Rosenthal compactum K is a pre-metric compactum of degree at most n, then one of the following alternatives holds:

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2. K contains copies of $S_n(I)$.
3. K contains copies of $D_n(2^{\mathbb{N}})$.

The space $S_n(I)$

Definition ($n-$Split Interval $S_n(I)$)

Fix $n \geq 2$ a natural number. We define the $n-$Split Interval $S_n(I)$ as the space $I \times \{0, \cdots, n-1\}$ endowed with the topology for which the points (x, i) con $i \in \{2, \cdots, n-1\}$ are isolated

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$$](y, 0), (x, 1)[\ \cup \ ]y, x[\times\{2, \cdots, n-1\} \text{ with } y \in I,\, y < x$$
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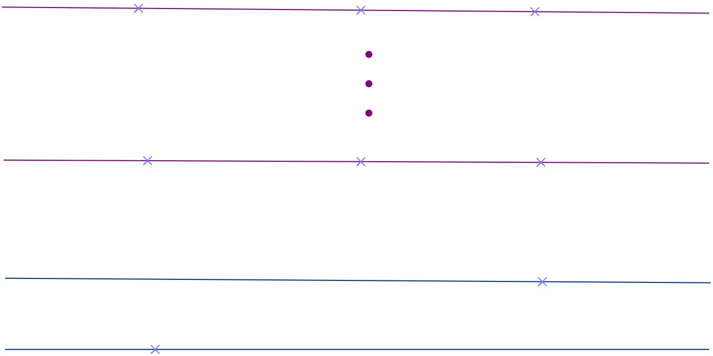
The topology of $S_n(I)$



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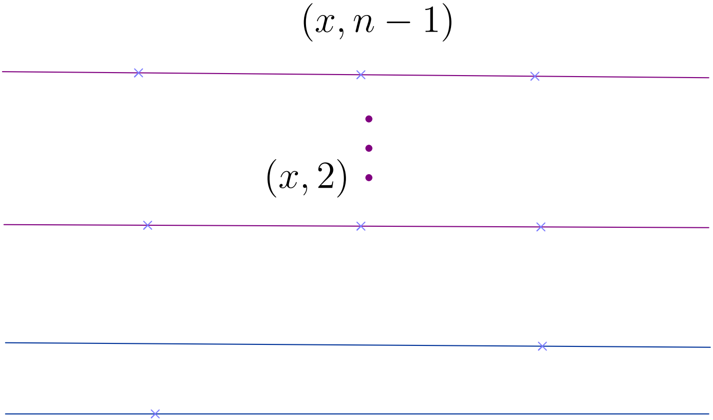

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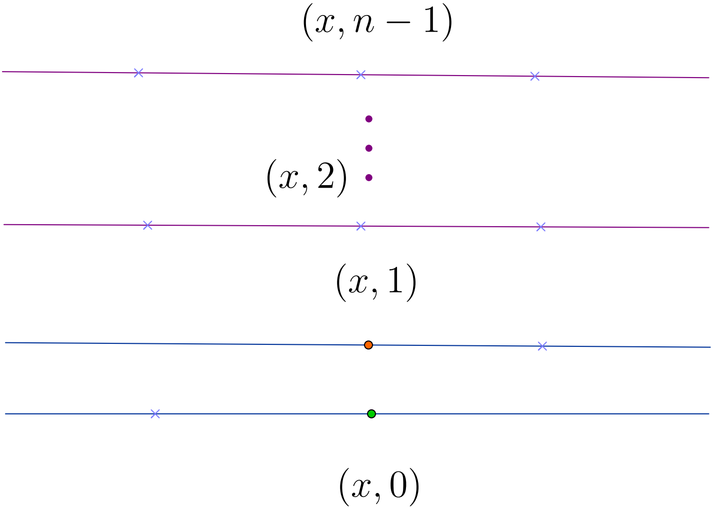

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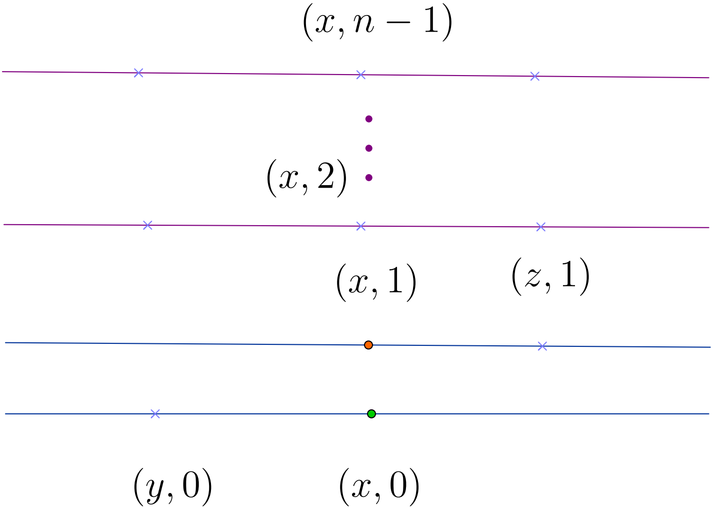

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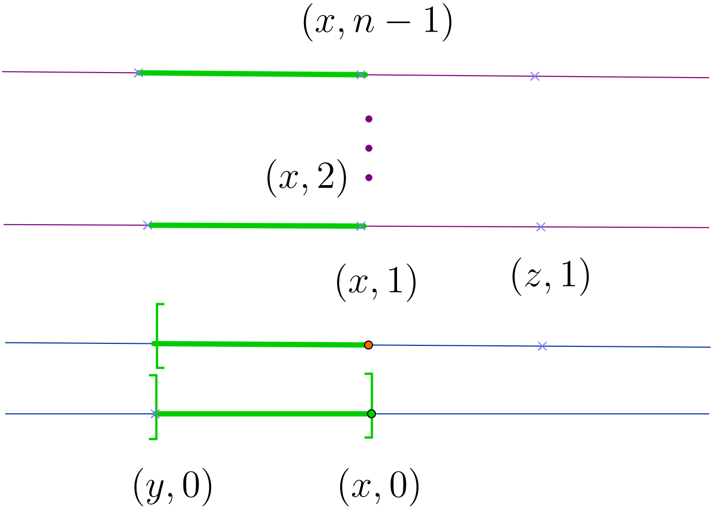

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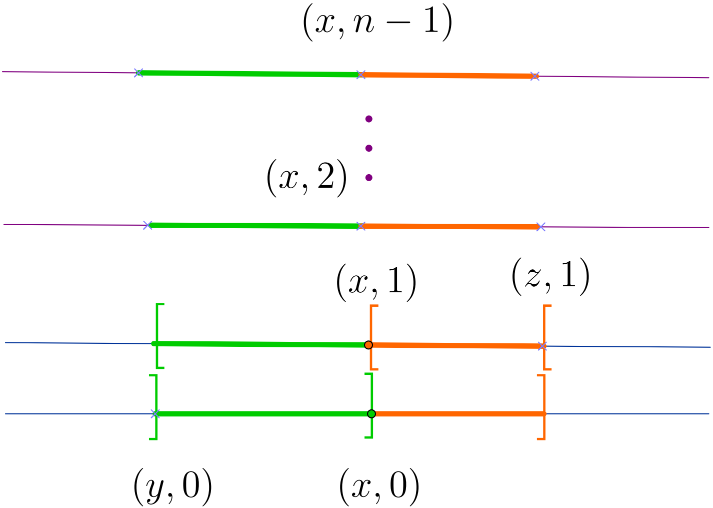

The space $D_n(2^{\mathbb{N}})$

Definition (Alexandroff $n-$plicate $D_n(X)$)

Given a topological Hausdorff space $(X, \mathscr{T})$ and $n \geq 2$ a natural number, we define the Alexandroff $n-$plicate $D_n(X)$ as the space $X \times \{0, \cdots, n-1\}$ endowed with the topology for which the points (x, i) with $i \in \{1 \cdots, n-1\}$ are isolated

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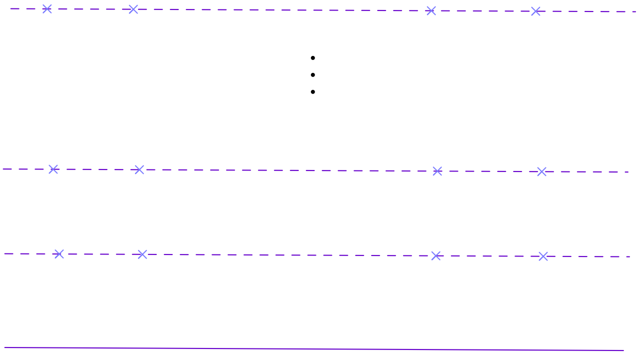
$$\mathscr{U} \times \{0, \cdots, n-1\} \setminus \bigcup_{i=1}^{n-1} \{(x, i)\}$$

where $\mathscr{U}$ is a neighbourhood of x in X.

We are only interested in $X = 2^{\mathbb{N}}$.

The topology of $D_n(2^{\mathbb{N}})$



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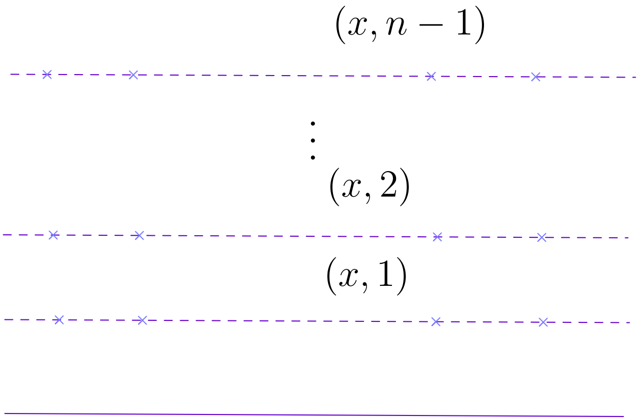

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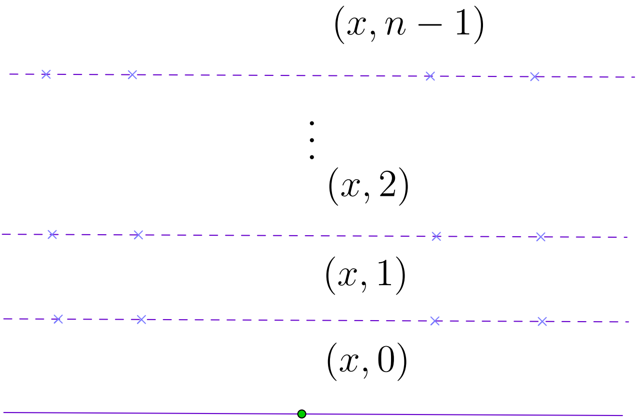

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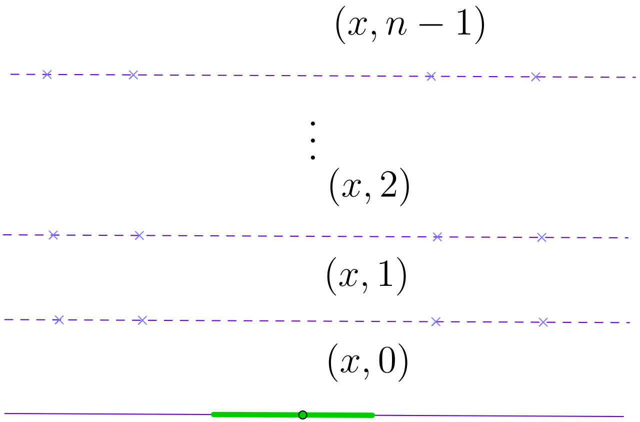

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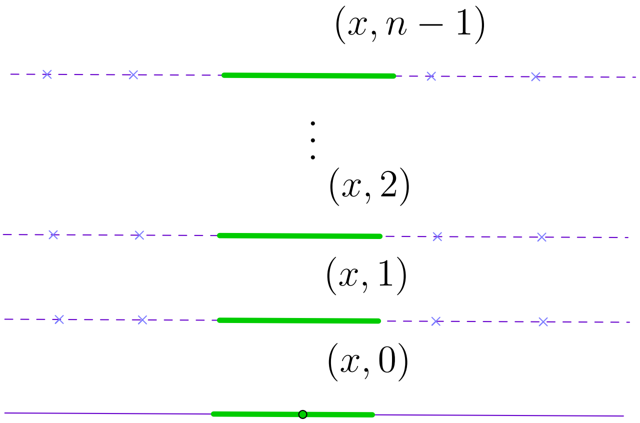

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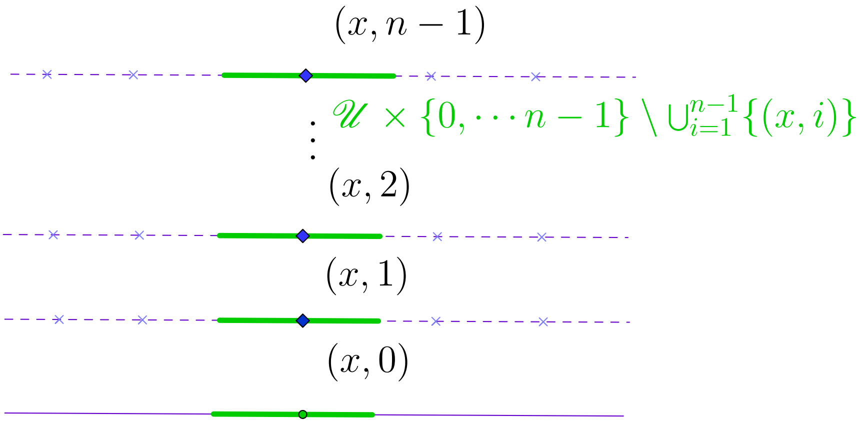

Final comments on $S_n(I)$ and $D_n(2^{\mathbb{N}})$

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$$\Downarrow$$

$S_n(I)$ and $D_n(2^{\mathbb{N}})$ minimal structures

The proof: Introduction

- Our proof uses some of the Todorcevic's original ideas [Tod99] to introduce $S_n(I)$ and $D_n(2^{\mathbb{N}})$ in K.

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We give only an sketch for $n = 3$

The proof: the ingredients.

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The proof: some preliminary comments.

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- We will use the following characterization for pre-metric compacta of degree at most n:

Proposition

Let X a set and K a Rosenthal compactum in $\mathbb{R}^X$. The compact space K is a pre-metric compactum of degree at most n iff there exists a countable subset D_0 of X for which the map

$$\begin{array}{cccc} \pi_{D_0}: & K & \longrightarrow & \{f \restriction_{D_0} : f \in K\} \\ & f & \mapsto & f \restriction D_0 \end{array}$$

is at most n to 1.

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$$+$$

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First step: A transfinite induction

$$D_0$$

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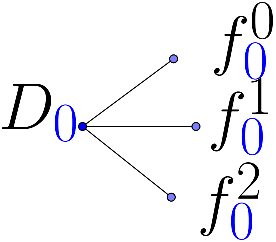

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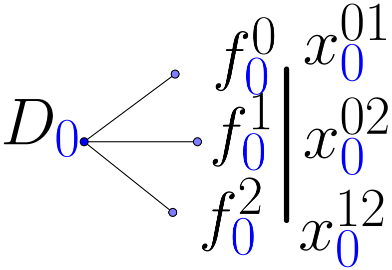
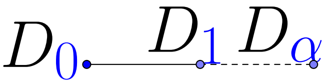


α th step of the induction

Given an ordinal $\alpha < \omega_1$

$$D_\alpha = D_0 \cup \{(x_\gamma^{01}, x_\gamma^{02}, x_\gamma^{12})\}_{\gamma<\alpha}$$



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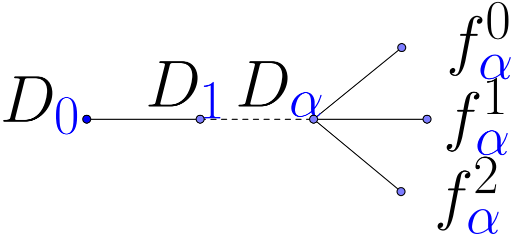

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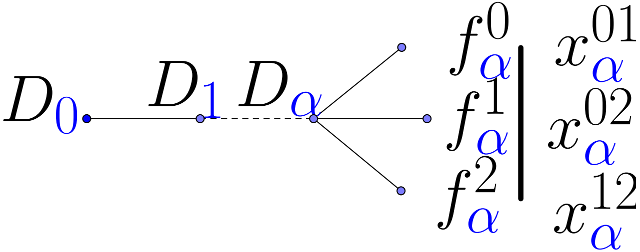

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(1) $f^i_\alpha \restriction_{D_0} = f^j_\alpha \restriction_{D_0}$.
(2) $f^i_\alpha(x^{ij}_\alpha) \neq f^j_\alpha(x^{ij}_\alpha)$
(3) $f^k_\alpha(x^{ij}_\beta) = f^l_\alpha(x^{ij}_\beta)$ for every $\beta < \alpha$.
(4) $f^i_\alpha \restriction_{D_0} \neq f^j_\beta \restriction_{D_0}$ when $\alpha \neq \beta$.

Second step: building a perfect tree

- We will denote on the sequel by h_α the common restriction to D_0 of the functions $f^0_\alpha, f^1_\alpha, f^2_\alpha$.

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The nodes of the tree

$\sigma \in 2^{<\mathbb{N}}$

$$(t_\sigma^0, t_\sigma^1)$$

$$\cap$$

$$(\mathbb{N}^{<\mathbb{N}})^3 \times \mathbb{N}^{<\mathbb{N}}$$

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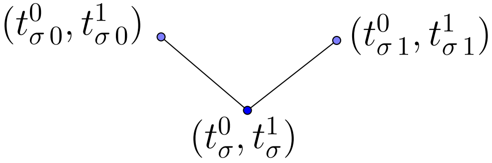

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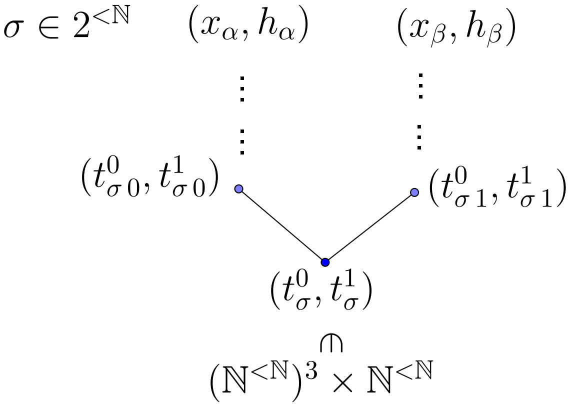

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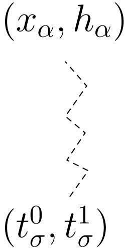
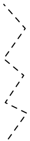


The nodes of the tree.

$$(x_\alpha, h_\alpha)$$



$$\Longrightarrow \quad \begin{array}{l} g^i_\sigma \approx f^i_\alpha \\ \pitchfork \\ K_0 \end{array}$$

$$(t^0_\sigma, t^1_\sigma)$$

Third step: Applying some Ramsey like results

Note that $\{g^i_\sigma\}_{\sigma\in 2^{<\mathbb{N}}}$ is a relative compact in $\boldsymbol{\mathcal{B}_1}(\mathbb{N}^{\mathbb{N}})$

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$$\{g^i_\sigma\}_{\sigma\in 2^{<\mathbb{N}}} \subset K$$

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$$\Downarrow \text{ (Ramsey [Tod99])}$$

Third step: Applying some Ramsey like results

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$$\Downarrow \text{ (Ramsey [Tod99])}$$

There exists a perfect subset $P \subset 2^\mathbb{N}$ and $\{m_k\}_{k\in\mathbb{N}}$ strictly increasing such that for every index i and $a \in P$,

$$g^i_a = \lim_k g^i_{a\upharpoonright m_k}$$

Third step: Applying some Ramsey like results

- Given $a \in P$ we denote by x_a the unique extension in $\mathbb{N}^{\mathbb{N}}$ of $\{t^0_{a \restriction m}\}_{m \in \mathbb{N}}$

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Given $a \in P$ and different indexes $i, j \in \{0, 1, 2\}$, we define

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Cases

1. There exists a unique index i such that g_a^i are not isolated from the other functions by the neighbourhoods $\mathscr{U}_j(g_a^i)$.

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Cases

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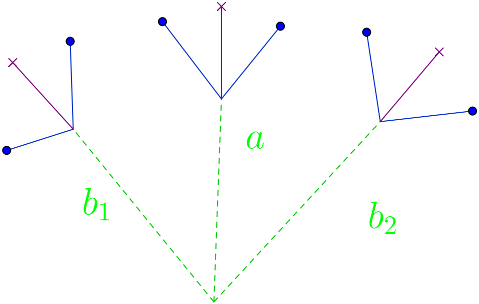
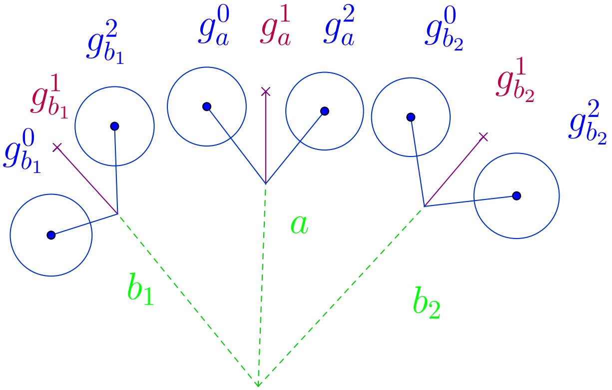
2. There exists a unique index i (resp. j) such that g_a^i (resp. g_a^j) are isolated from those functions which are on their right (resp. left) sides by the neighbourhoods $\mathscr{U}_j(g_a^i)$.

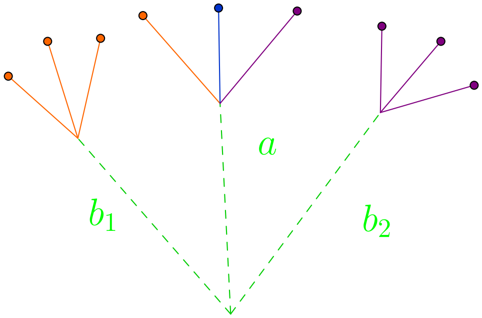
Case 1 with $i = 1$



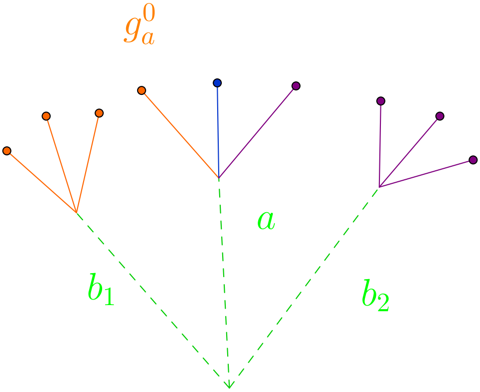
Case 1 with $i = 1$



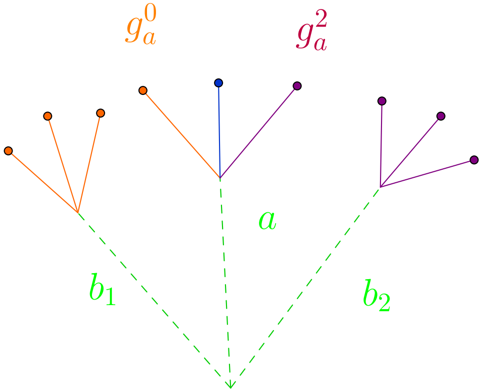
Case 2 with $i = 0$ and $j = 2$



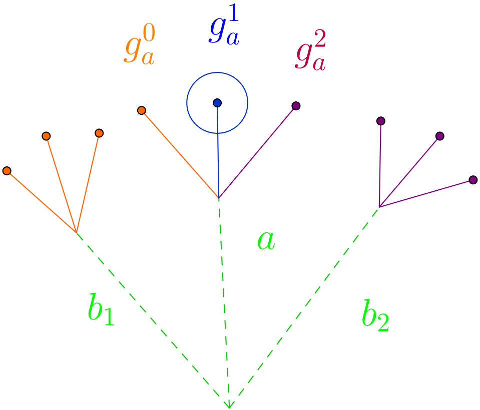
Case 2 with $i = 0$ and $j = 2$



Case 2 with $i = 0$ and $j = 2$



Case 2 with $i = 0$ and $j = 2$

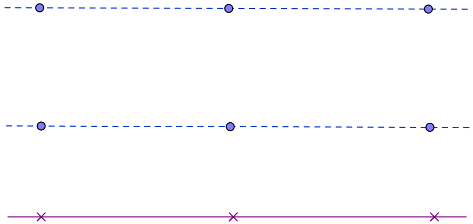


Fourth and last step: Introducing $D_3(2^{\mathbb{N}})$ and $S_3(I)$ inside K

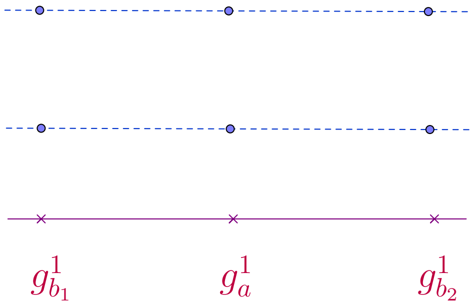
Note the parallelism between

- Case 1 and $D_3(2^{\mathbb{N}})$.

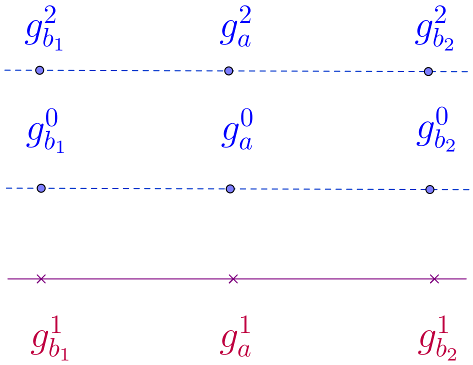
Case 1 and $D_3(2^{\mathbb{N}})$



Case 1 and $D_3(2^{\mathbb{N}})$



Case 1 and $D_3(2^{\mathbb{N}})$

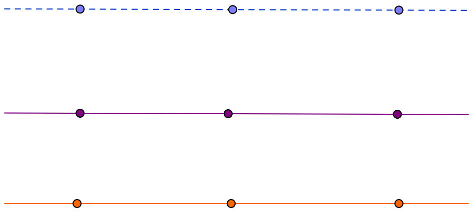


Fourth and last step: Introducing $D_3(2^{\mathbb{N}})$ and $S_3(I)$ inside K

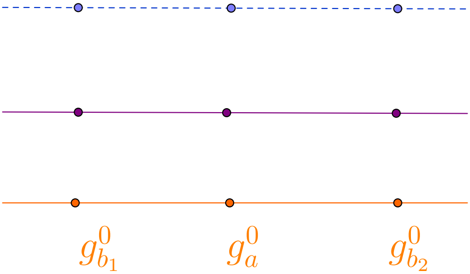
Note the parallelism between

- Case 1 and $D_3(2^{\mathbb{N}})$.
- Case 2 and $S_3(I)$.

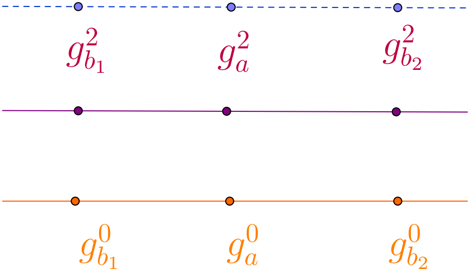
Case 2 and $S_3(I)$



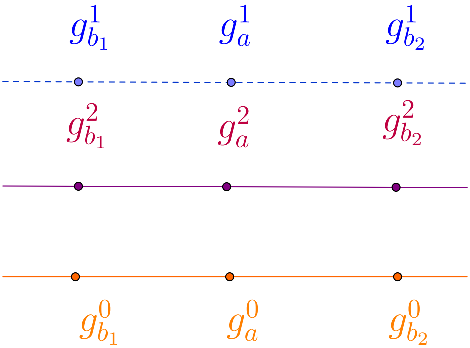
Case 2 and $S_3(I)$



Case 2 and $S_3(I)$



Case 2 and $S_3(I)$



Thank you very much for your assistance and for your patience (which is $> \mathfrak{c}$)!

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