Unions of regular families

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Cardinal coefficients

Let $X$ - Polish space, $I \subseteq P(X)$-σ ideal on $X$.

\[ \text{cov}(I) = \min \{|A| : A \subseteq I \land \bigcup A = X\} \]

\[ \text{cov}_h(I) = \min \{|A| : A \subseteq I \land (\exists B \in \text{Bor}(X) \setminus I) B \subseteq \bigcup A\} \]

$\mathcal{N}$ - all null subsets of $\mathbb{R}$ and $\mathcal{M}$ all meager subsets of $X$. 
Bukovský Theorem (1979)

- For every partition $\mathcal{A}$ of real line onto null sets there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is not Lebesgue measurable set of $\mathbb{R}$.
- For every partition $\mathcal{A}$ of real line onto meager sets there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ does not have Baire property.

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- For every partition $\mathcal{A}$ of real line onto meager sets there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ does not Baire property.

Let $X$ – uncountable Polish space $I \subseteq P(X)$ $\sigma$-ideal on $X$ with Borel base containing all singletons of $X$. Then $(X, I)$ - Polish ideal space.

$A \subseteq X$ is $I$-measurable if $A \in \text{Bor}(X)[I]$ i.e

$$A = B \Delta I$$ for some $B \in \text{Bor}(X)$ and $I \in I$.

$\mathcal{A} \subseteq P(X)$ is summable if for any $\mathcal{A}' \subseteq \mathcal{A}$, $\bigcup \mathcal{A}'$ is $I$-measurable set.

$\mathcal{A} \subseteq P(X)$ is point-finite family if

$$(\forall x \in X) \ |\{A \in \mathcal{A} : x \in A\}| < \aleph_0.$$
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Polish ideal space

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Theorem (Brzuchowski, Cichoń, Grzegorek and Ryll-Nardzewski (1979))

Assume that \((X, I)\) is Polish ideal space. If \(A \subseteq I\) is point-finite family such that \(\bigcup A = X\) then there is a \(A' \subseteq A\) such that \(\bigcup A'\) is not \(I\)-measurable in \(X\).

Theorem (Fremlin)

*It is relative consistent with ZFC that there exists a $\mathcal{N}$-summable point-countable family $\mathcal{A} \subseteq \mathcal{N}$ of $[0,1]$ s.t. $\bigcup \mathcal{A} = [0,1]$.*


Theorem (Fremlin-Todorcević)

*Let $\mathcal{A} \subseteq \mathcal{N}$ be a partition of $[0,1]$, then for every $\epsilon > 0$ there is a $\mathcal{A}' \subseteq \mathcal{A}$ such that $1 - \epsilon < \lambda^*(\bigcup \mathcal{A}')$ and $\lambda_*(\bigcup \mathcal{A}') < \epsilon$.*

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Cantor-Bendixon rank

For topological space $X$ let $A \subseteq X$ and $A'$ is a set of all accumulation points of $A$. For any $\alpha \in ON$

- $A^{(\alpha+1)} = (A^{(\alpha)})'$
- $\alpha$-limit then $A^\alpha = \bigcap_{\xi<\alpha} A^{(\xi)}$

Theorem (C-M-R-CRN-Z)

Let $(X, I)$ Polish ideal space. Assume that $A \subseteq P(X)$ is a family

- every $A \in A$ is closed of $X$,
- $(\exists \alpha \in \omega_1)(\forall A \in A) \ A^{(\alpha)} = \emptyset$,
- $A$ is $I$-summable family.

Then $\bigcup A \in I$.

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Then $\bigcup A \in I$.

Let \((X, I)\) be a Polish ideal space. Then \(A \subseteq X\) is completely \(I\)-nonmeasurable set if

\[(\forall B \in \text{Bor}(X) \setminus I) \ B \cap A \neq \emptyset \land B \cap A^c \neq \emptyset\]

- every completely \([X]^{\omega}\) -nonmeasurable set is a Bernstein set,
- every completely \(\mathcal{N}\) -nonmeasurable set \(A \subseteq [0, 1]\) has \(\lambda_*(A) = 0\) and \(\lambda^*(A) = 1\).
- every completely \(\mathcal{M}\) -nonmeasurable set has not Baire property in each nonempty open set of \(X\).
Completely nonmeasurable set

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Theorem (C-M-R-CRN-\(\breve{Z}\))

Let \((X, I)\) be Polish ideal space. Let \(A \subseteq I\) be a family such that:

1. \(\bigcup A = X\),
2. for every \(x \in X\) we have \(\bigcup \{A \in A : x \in A\} \in I\),
3. \(\text{cov}_h(I) = \mathfrak{c}\),

then there is \(A' \subseteq A\) s.t. \(\bigcup A'\) is completely \(I\)-nonmeasurable in \(X\).

Theorem (C-M-R-CRN-\(\breve{Z}\))

Let \(A \subseteq \mathcal{M}\) be a partition of \(\mathbb{R}\) then there is \(A' \subseteq A\) s.t. \(\bigcup A'\) is completely \(\mathcal{M}\)-nonmeasurable set in \(\mathbb{R}\).
Theorem (C-M-R-CRN-Ż)
Let $(X, I)$ be Polish ideal space. Let $A \subseteq I$ be a family such that:
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Theorem (C-M-R-CRN-Ż)
Let $A \subseteq \mathcal{M}$ be a partition of $\mathbb{R}$ then there is $A' \subseteq A$ s.t. $\bigcup A'$ is completely $\mathcal{M}$-nonmeasurable set in $\mathbb{R}$.
We say that uncountable $\kappa$ is quasi-measurable if there exists $\kappa$-additive c.c.c. ideal $I \subseteq P(\kappa)$ (i.e. $P(\kappa)/I$ is c.c.c. algebra).

**Theorem (Żeberski, RR)**

Assume that $(X, I)$ is Polish ideal space and $I$ is c.c.c. and there is no quasi-measurable $\kappa \leq \mathfrak{c}$. Then for every point-finite $\mathcal{A} \subseteq I$ such that $\bigcup \mathcal{A} = X$ there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is completely $I$-nonmeasurable in $X$. 
Non existence of small quasi-measurable cardinals

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Assume that $(X, I)$ is Polish ideal space and $I$ is c.c.c. and there is no quasi-measurable $\kappa \leq c$. Then for every point-finite $\mathcal{A} \subseteq I$ such that $\bigcup \mathcal{A} = X$ there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is completely $I$-nonmeasurable in $X$. 
Definition
Let $X$ Polish space then every partition $\pi \subseteq P(X)$ of $X$ is strongly Borel measurable if for every closed set $D \subseteq X$

$$\bigcup\{A \in \pi : A \cap D \neq \emptyset\} \in \text{Bor}(X).$$

For $F \subseteq X \times Y$ and $(x, y) \in X \times Y$ we define

$$F_x = \{t \in Y : (x, t) \in F\}, \quad F^y = \{s \in X : (s, y) \in F\}, \quad \pi_X(F) = \bigcup\{F_y : y \in Y\}.$$
Theorem (Żeberski, RR)

Let \((X, I)\) Polish ideal space s.t. each \(B \in Bor(X) \setminus I\) contains \(I\)-positive perfect set. Then for every strongly Borel partition \(A \subseteq I\) of \(X\) there is \(A'\) s.t. \(\bigcup A'\) is completely \(I\)-nonmeasurable.

Theorem (Żeberski and RR)

Let \((X, I)\) Polish ideal space, \(Y\) topological space. Assume that \(f : X \to Y\) is \(I\)-measurable map s.t. for any \(y \in Y\), \(f^{-1}[\{y\}] \in I\). Then there is \(T \subseteq Y\) s.t. \(f^{-1}[T]\) is completely \(I\)-nonmeasurable.

Theorem (Żeberski, RR)

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Theorem (Żeberski, RR)

Assume that $(X, I)$ Polish ideal space, $I$ is c.c.c. and $Y$ topological space, $F : X \to Y$ is $I$-measurable multifunction s.t. for any $x \in X$ $f(x)$ is finite. Then there exists $T \subseteq Y$ s.t. $f^{-1}[T]$ is completely $I$-nonmeasurable.

Theorem (Żeberski, RR)

Assume that $(X, I)$ Polish ideal space, $I$ is c.c.c. and $Y$ topological space. Let $F \subseteq X \times Y$ analytic relation s.t.

1. $(\forall y \in Y) \ F^y \in I,$
2. $X \setminus \pi_X[F] \in I,$
3. $(\forall x \in X) \ |F_x| < \aleph_0,$

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Thank You for your attention