Unions of regular families

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Cardinal coefficients

Let X - Polish space, $I \subseteq P(X)$-σ ideal on X.

$$cov(I) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq I \wedge \bigcup \mathcal{A} = X\}$$

$$cov_h(I) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq I \wedge (\exists B \in Bor(X) \setminus I) B \subseteq \bigcup \mathcal{A}\}$$

$\mathscr{N}$ - all null subsets of $\mathbb{R}$ and $\mathscr{M}$ all meager subsets of X.

Bukovsky Theorem (1979)

- For every partition $\mathcal{A}$ of real line onto null sets there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is not Lebesgue measurable set of $\mathbb{R}$.
- For every partition $\mathcal{A}$ of real line onto meager sets there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ does not Baire property.

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Polish ideal space

Let X – uncountable Polish space $I \subseteq P(X)$ σ-ideal on X with Borel base containing all singletons of X.
Then (X, I) - Polish ideal space.
$A \subseteq X$ is I-measurable if $A \in Bor(X)[I]$ i.e

$$A = B \triangle I \text{ for some } B \in Bor(X) \text{ and } I \in I.$$

$\mathcal{A} \subseteq P(X)$ is summable if for any $\mathcal{A}' \subseteq \mathcal{A}$, $\bigcup \mathcal{A}'$ is I-measurable set.
$\mathcal{A} \subseteq P(X)$ is point-finite family if

$$(\forall x \in X)\ |\{A \in \mathcal{A} :\ x \in A\}| < \aleph_0.$$

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Theorem (Brzuchowski, Cichoń, Grzegorek and Ryll-Nardzewski (1979))

Assume that (X, I) is Polish ideal space. If $\mathcal{A} \subseteq I$ is point-finite family such that $\bigcup \mathcal{A} = X$ then there is a $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is not I-measurable in X.

J.Brzuchowski J. Cichoń E. Grzegorek C. Ryll-Nardzewski, On the existence of nonmeasurable unions, Bull. Polish Acad. Sci. Math. 27 (1979) 447–448.

Theorem (Fremlin)

It is relative consistent with ZFC that there exists a $\mathscr{N}$*-summable point-countable family* $\mathcal{A} \subseteq \mathscr{N}$ *of* $[0,1]$ *s.t.* $\bigcup \mathcal{A} = [0,1]$.

D. Fremlin, Measure-additive coverings and measurable selectors, Dissertationes Math. 260 (1987).

Theorem (Fremlin-Todorcević)

Let $\mathcal{A} \subseteq \mathscr{N}$ *be a partition of* $[0,1]$*, then for every* $\epsilon > 0$ *there is a* $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $1 - \epsilon < \lambda^*(\bigcup \mathcal{A}')$ *and* $\lambda_*(\bigcup \mathcal{A}') < \epsilon$.

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Cantor-Bendixon rank

For topological space X let $A \subseteq X$ and A' is a set of all accumulation points of A. For any $\alpha \in ON$

- $A^{(\alpha+1)} = (A^{(\alpha)})'$
- α-limit then $A^{\alpha} = \bigcap_{\xi<\alpha} A^{(\xi)}$

Theorem (C-M-R-CRN-Z)

Let (X, I) Polish ideal space. Assume that $\mathcal{A} \subseteq P(X)$ is a family

- *every $A \in \mathcal{A}$ is closed of X,*
- $(\exists \alpha \in \omega_1)(\forall A \in \mathcal{A})\ A^{(\alpha)} = \emptyset$,
- *$\mathcal{A}$ is I-summable family.*

Then $\bigcup \mathcal{A} \in I$.

J. Cichoń, M. Morayne, R. Rałowski, C. Ryll-Nardzewski, S. Żeberski, On nonmeasurable unions, Topol. and its Appl. 154 (2007) 884-893

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Completely nonmeasurable set

Let (X, I) be a Polish ideal space. Then $A \subseteq X$ is completely I-nonmeasurable set if

$$(\forall B \in Bor(X) \setminus I)\ B \cap A \neq \emptyset \wedge B \cap A^c \neq \emptyset$$

- every completely $[X]^\omega$-nonmeasurable set is a Bernstein set,
- every completely $\mathscr{N}$-nonmeasurable set $A \subseteq [0,1]$ has $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$.
- every completely $\mathscr{M}$-nonmeasurable set has not Baire property in each nonempty open set of X.

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Theorem (C-M-R-CRN-Ż)

Let (X, I) be Polish ideal space. Let $\mathcal{A} \subseteq I$ be a family such that:

1. $\bigcup \mathcal{A} = X$,
2. *for every $x \in X$ we have $\bigcup\{A \in \mathcal{A} : x \in A\} \in I$,*
3. $cov_h(I) = \mathfrak{c}$,

then there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is completely I-nonmeasurable in X.

Theorem (C-M-R-CRN-Ż)

Let $\mathcal{A} \subseteq \mathscr{M}$ be a partition of $\mathbb{R}$ then there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is completely $\mathscr{M}$-nonmeasurable set in $\mathbb{R}$.

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Non existence of small quasi-measurable cardinals

We say that uncountable κ is quasi-measurable if there exists κ-additive $c.c.c.$ ideal $I \subseteq P(\kappa)$ (i.e. $P(\kappa)/I$ is c.c.c. algebra).

Theorem (Żeberski, RR)

Assume that (X, I) is Polish ideal space and I is c.c.c. and there is no quasi-measurable $\kappa \leq \mathfrak{c}$. Then for every point-finite $\mathcal{A} \subseteq I$ such that $\bigcup \mathcal{A} = X$ there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is completely I-nonmeasurable in X.

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Definiton

Let X Polish space then every partition $\pi \subseteq P(X)$ of X is **strongly Borel measurable** if for every closed set $D \subseteq X$

$$\bigcup\{A \in \pi : \ A \cap D \neq \emptyset\} \in Bor(X).$$

For $F \subseteq X \times Y$ and $(x, y) \in X \times Y$ we define

$$F_x = \{t \in Y : \ (x, t) \in F\}, \quad F^y = \{s \in X : \ (s, y) \in F\},$$

$$\pi_X(F) = \bigcup\{F_y : \ y \in Y\}.$$

Theorem (Żeberski, RR)

Let (X, I) Polish ideal space s.t. each $B \in Bor(X) \setminus I$ contains I-positive perfect set. Then for every strongly Borel partition $\mathcal{A} \subseteq I$ of X there is $\mathcal{A}'$ s.t. $\bigcup \mathcal{A}'$ is completely I-nonmeasurable.

Theorem (Żeberski and RR)

Let (X, I) Polish ideal space, Y topological space. Assume that $f : X \to Y$ is I-measurable map s.t. for any $y \in Y$, $f^{-1}[\{y\}] \in I$. Then there is $T \subseteq Y$ s.t. $f^{-1}[T]$ is completely I-nonmeasurable.

R. Rałowski and S. Żeberski, Complete nonmeasurability in regular families, Houston Journal of Mathematics, vol 34 no 3, (2008)

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Theorem (Żeberski, RR)

Assume that (X, I) Polish ideal space, I is c.c.c. and Y topological space, $F : X \to Y$ is I-measurable multifunction s.t. for any $x \in X$ $f(x)$ is finite. Then there exists $T \subseteq Y$ s.t. $f^{-1}[T]$ is completely I-nonmeasurable.

Theorem (Żeberski, RR)

Assume that (X, I) Polish ideal space, I is c.c.c. and Y topological space. Let $F \subseteq X \times Y$ analytic relation s.t.

1. $(\forall y \in Y)\ F^y \in I$,
2. $X \setminus \pi_X[F] \in I$,
3. $(\forall x \in X)\ |F_x| < \aleph_0$,

then there exists $T \subseteq Y$ s.t. $F^{-1}[T]$ is completely I-nonmeasurable.

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Thank You for your attention