Some notes concerning lifting arguments

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1. THE SACKS-LIKE GENERICS.

Notation: j will denote a non-trivial elementary extender ultrapower embedding with a critical point κ as detailed below, unless stated otherwise. **We will assume GCH.**

Recall that $j : V \to M$ is an *extender ultrapower* if

$$M = \{j(f)(\alpha) \,|\, f : \kappa \to V, \alpha < \lambda\},$$

for some $\lambda > \kappa$. In what follows think of λ as κ^{++} of V. We will assume $H(\kappa^{++}) \subseteq M$, so that $\kappa^{++} = (\kappa^{++})^M < j(\kappa) < \kappa^{+3}$, and all M-cardinals regular in M in the interval $((\kappa^{+3})^M, j(\kappa))$ have V-cofinality κ^+. Morever, $V \cap {}^\kappa M \subseteq M$.

Fact 1 (Lifting lemma, Silver) *Let* $j : V \to M$ *be elementary and* $\mathbb{P}$ *a forcing notion. If* G *is* $\mathbb{P}$*-generic and* H *is* $j(\mathbb{P})$*-generic over* M *and* $j[G] \subseteq H$*, then* j *lifts to an elementary* $j^* : V[G] \to M[H]$ *such that* $j^* \restriction V = j$ *and* $j^*(G) = H$*. If moreover* j *was an extender embedding, so is* j^**.*

Proof. For $f : \kappa \to V$ which takes its range in $\mathbb{P}$-names, define $f^* : \kappa \to V[G]$ by $f^*(\alpha) = (f(\alpha))^G$. Then $M[H] = \{j(f^*)(\alpha) \mid f : \kappa \to V, \alpha < \lambda\}$. $\square$

Let X be a subset of a forcing notion $\mathbb{P}$. We write $\mathscr{G}(X)$ for the upper closure of X, i.e.

$$\mathscr{G}(X) = \{q \in \mathbb{P} \mid (\exists p \in X)\ p \leq q\}.$$

Fact 2 *Let $j : V \to M$ be an extender ultrapower embedding and $\mathbb{P}$ a κ^+-distributive forcing notion in V and let G be $\mathbb{P}$-generic. Then $\mathscr{G}(j[G])$ is $j(\mathbb{P})$-generic over M. By Lifting lemma, j lifts to $V[G] \to M[\mathscr{G}(j[G])]$.*

We wish to formulate an "as good as possible" variation of Fact 2 for forcings $\mathbb{P}$ which add new subsets of κ. Typically, $\mathbb{P}$ would be κ-closed and satisfy some sort of κ-fusion.

Definition 3 *Let $\mathbb{P}$ be a α^+-closed (separative) forcing notion for some regular cardinal α. For a non-empty $X \subseteq \mathbb{P}$, we denote by $\mathscr{G}_\alpha(X)$ the collection of all conditions p in $\mathbb{P}$ such that there exists a $\leq$-decreasing α-sequence $\langle p_i \,|\, i < \alpha \rangle$ of elements in X satisfying $\bigwedge_{i<\alpha} p_i \leq p$. We say that $\mathscr{G}_\alpha(X)$ is α-generated by X.*

Example. Let G be $\mathbb{P}$-generic for an α^+-closed forcing $\mathbb{P}$. Then (in $V[G]$) $\mathscr{G}_\alpha(G) = G$. [It suffices to argue that if p_i for $i < \alpha$ are decreasing and in G, then $\bigwedge_{i<\alpha} p_i$ is in G. Notice that $D = \{p \,|\, p \leq \bigwedge_{i<\alpha} p_i\} \cup \{p \,|\, (\exists j < \alpha) p \perp p_j\}$ is dense, which implies the claim.]

Let $\mathbb{P} = (\mathbb{P}_\alpha, \dot{Q}_\alpha)_{\alpha \leq \kappa}$ be a forcing iteration on inaccessibles $\alpha \leq \kappa$ (typically with Easton support), where each $\dot{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be α-closed. Let $G * g$ be $\mathbb{P}_\kappa * \dot{Q}_\kappa$ generic, and let us write $Q = (\dot{Q}_\kappa)^G$. Assume we have lifted $j : V \to M$ in $V[G * g]$ to

$$j^* : V[G] \to M[G * g * H],$$

where H is the "middle generic" for the forcing $j(\mathbb{P})$ in the interval $(\kappa, j(\kappa))$. Assume further that $V[G * g] \cap {}^\kappa M[G * g * H] \subseteq M[G * g * H]$. (Note that this is a typical situation in forcing arguments, and one which is relatively easy to get; for instance when $\mathbb{P}$ is κ^+-cc, or $\dot{Q}_\kappa$ is the κ-Sacks.)

Let us write j instead of j^* in what follows.

Lemma 4 *$\mathscr{G}_\kappa(j[g])$ is a filter on $j(Q)$ which contains $j[g]$.*

In particular, if $\mathscr{G}_\kappa(j[g])$ happens to hit all dense open sets in $j(Q)$ in $M[G * g * H]$, then j lifts (in $V[G * g]$) to $j : V[G * g] \to M[G * g * H * \mathscr{G}_\kappa(j[g])]$.

Proof. Since Q is κ-closed, $j(Q)$ is κ^+-closed in $V[G * g]$. It suffices to show that if $\langle p_i \mid i < \kappa \rangle$ and $\langle q_i \mid i < \kappa \rangle$ are decreasing sequences of elements in $j[g]$, then we can find a decreasing sequence of elements in $j[g]$ $\langle r_i \mid i < \kappa \rangle$ such that $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<j} p_i$ and $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<j} q_i$. We define the sequence $\langle r_i \mid i < j \rangle$ by induction, getting below p_i and q_i at stage i (p_i and q_i are compatible by elementarity), and taking infima at limits. □

The filter $\mathscr{G}_\kappa(j[g])$ is the least possible:

Lemma 5 *Assume h is an $M[G*g*H]$-generic filter for $j(Q)$ which exists in $V[G*g]$ and contains $j[g]$, then $\mathscr{G}_\kappa(j[g]) \subseteq h$. It follows that if $\mathscr{G}_\kappa(j[g])$ is generic, then it is the unique generic in $V[G*g]$ which contains $j[g]$.*

Proof. Let us denote $M^* = M[G*g*H*h]$. Since h is a generic filter for a $j(\kappa)$-closed forcing in $M[G*g*H]$, by the above example $(\mathscr{G}_{j(\kappa)}(h))^{M^*}$ (let us denote as as h^*) is included in h. Since M^* contains as elements all κ-sequences of its elements which are available in $V[G*g]$, $\mathscr{G}_\kappa(j[g])$ (which is the same whether taken in $V[G*g]$ or in M^*) is included in h^*, and so in h. $\square$

Let Q be the κ-Sacks forcing S, and $g \subseteq S$ a generic. Since S is κ-closed, $\mathscr{G}_\kappa(j[g])$ is a filter. Is it a generic filter?

- No, with the usual definition of Sacks(κ).

- Yes, if we allow the trees to split only at *non-regular* points (i.e. if $t \in T$ has regular length, then t does not split in T, even if splitting nodes are unbounded below t).

This also extends to iterations and more complicated forcings. In fact, except for the case when one specifically wishes to control the number of the liftings (see the paper by Sy Friedman and M. Magidor on the number of normal measures), "singular splitting" Sacks trees can be used in joint results with Sy Friedman by K. Thompson, N. Dobrinen, L. Zdomsky and myself.

2. SOME EXAMPLES.

Two examples of forcings with “strange” properties (in lifting contexts).

Example 1. (Relevant for iteration of Prikry-type forcings.)

For which Easton functions F (from REG to CARD) can one achieve the following:

$$\begin{aligned}(V, F, E, P) \models\ & GCH \wedge F \text{ is an Easton function with } P \subseteq CL(F) \wedge \\ & E \text{ is the class of regular cardinals} \wedge \\ & P \text{ are } F(\kappa)\text{-strong cardinals with } \kappa^+ < F(\kappa),\end{aligned}$$

then there is a cardinal-preserving generic extension W such that

$$\begin{aligned}(W, F, E, P) \models\ & F \text{ is the continuum function on } E \wedge \\ & \text{All elements in } P \text{ are sing strong limit card of cof } \omega.\end{aligned}$$

Assume a strong limit singular cardinal κ has cofinality ω in V^* and $\kappa^+ < 2^\kappa$ in V^*. Assume further that GCH holds below κ. Note that if the κ_i's are cofinal in κ for $i < \omega$, then the size of $\prod_{i<\omega} \kappa_i$ is $\kappa^\omega = 2^\kappa$. (Use extender-based Prikry forcing to get this situation).

Goal. Suppose we wish to add a single new subset to all regular cardinals $\alpha < \kappa$. Can we do this without collapsing cardinals? Products or iterations with direct limit at κ are easily seen to collapse κ to ω. So we will look at full support iterations/products.

If μ is a regular cardinal we write $\mathrm{Add}(\mu, 1)$ for the Cohen forcing adding a new subset of μ. Conditions in $\mathrm{Add}(\mu, 1)$ will be construed as defined on initial segments of μ (i.e., on ordinals less than μ) with range included in $\{0, 1\}$.

Definition 6 *Under this notion, we say that* p *in* $\mathrm{Add}(\mu, 1)$*, or more generally a generic for* $\mathrm{Add}(\mu, 1)$, codes $\delta < \mu$ at position $\delta' < \mu$ *if* p *restricted to* $[\delta', \delta' + \delta + 1)$ *is a sequence of* 1*'s followed by one* 0*, i.e., the* 1*'s starting at* δ' *have order type* δ *and this segment is terminated by* 0 *to determine which ordinal is being coded.*

By pcf results, there exists an increasing sequence $\langle \lambda_n \,|\, n < \omega \rangle$ of regular cardinals with limit κ such that there is a sequence $\langle f_\xi \,|\, \xi < \kappa^+ \rangle$ of elements in $\prod_{n<\omega} \lambda_n$ such that $\langle f_\xi \,|\, \xi < \kappa^+ \rangle$ is $<_{\mathsf{FIN}}$-cofinal in $\prod_{n<\omega} \lambda_n$ modulo the ideal of finite sets FIN.

Observation 7 *Let $\mathbb{P} = \prod_{i<\omega}^{\mathsf{FULL}} \mathsf{Add}(\lambda_i, 1)$ be the product of Cohen forcings with full support. Then $\mathbb{P}$ collapses 2^κ to κ^+.*

Proof. Let G be generic for $\mathbb{P}$, with g_i for $i < \omega$ generics for Add($\lambda_i, 1$)'s. Assume for simplicity first that $\langle f_\xi \,|\, \xi < \kappa^+ \rangle$ is cofinal on all coordinates, i.e., we do not allow the error modulo FIN. We define a function $h : \kappa^+ \to 2^\kappa = |\prod_{i<\omega} \lambda_i|$ which is onto as follows. For $\xi < \kappa^+$ let $h(\xi)$ be the sequence of ordinals $\langle \alpha_i \,|\, i < \omega \rangle$ in $\prod_{i<\omega} \lambda_i$ such that α_i is coded as in Definition 6 by g_i at the position $f_\xi(i)$ for each $i < \omega$.

We argue that h is onto. Let a sequence $s = \langle \beta_i \,|\, i < \omega \rangle$ in $\prod_{i<\omega} \lambda_i$ be given. By cofinality of $\langle f_\xi \,|\, \xi < \kappa^+ \rangle$, it is easy to see that the following set is dense:

$$D_s = \{p \in \mathbb{P} \,|\, \exists \xi < \kappa^+, p \text{ pointwise codes } s \text{ at places } f_\xi(i) \text{ for } i < \omega\}.$$

Now we rectify the argument to account for $<_{\mathsf{FIN}}$-cofinality. Define $h^* : \kappa^+ \to \prod_{i<\omega} \lambda_i$ as follows: let $h^*(\xi)$ be the family of all sequences $\langle \alpha_i \,|\, i < \omega \rangle$ such that $\langle \alpha_i \,|\, i < \omega \rangle$ is coded by g_i at position $f_\xi(i)$ for all but finitely many i's.

Note that the size of $h^*(\xi)$ is κ for every $\xi < \kappa^+$. The observation follows once we show that $\prod_{i<\omega} \lambda_i$ is covered by the union $\bigcup_{\xi<\kappa^+} h^*(\xi)$.

If p is a condition in $\mathbb{P}$ let $\mathsf{Code}_\xi(p)$ denote the family of sequences $\langle \alpha_i \,|\, i < \omega \rangle$ such that $p(i)$ codes α_i at the position $f_\xi(i)$ for all but finitely many i's. Let a sequence $s = \langle \beta_i \,|\, i < \omega \rangle$ in $\prod_{i<\omega} \lambda_i$ be given. By $<_{\mathsf{FIN}}$-cofinality of $\langle f_\xi \,|\, \xi < \kappa^+ \rangle$, it is easy to see that the following set is dense:

$$D_s = \{p \in \mathbb{P} \,|\, \exists \xi < \kappa^+, s \in \mathsf{Code}_\xi(p)\}.$$

This proves the observation.

Example 2. (A problem) In papers (joint with Sy Friedman) concerning realisation of Easton functions, the following issue was unresolved.

Question. (A special case) (GCH) Starting from an embedding $j : V \to M$ such that M is closed under κ-sequences and $\kappa^{++} = (\kappa^{++})^M < j(\kappa) < \kappa^{+3}$, can you force $2^\alpha = \alpha^{++}$ for every $\alpha \leq \kappa$ such that α is either inaccessible, or a double successor of an inaccessible and preserve κ's measurability?

The initial assumption about j has the consistency strength of $o(\kappa) = \kappa^{++}$, which is (by work of M. Gitik and others) an optimal strength for the failure of GCH at a measurable.

Let $\mathbb{P} = (\mathbb{P}_\alpha, \dot{Q}_\alpha)_{\alpha \leq \kappa}$ be an iteration (with Easton support), where for each $\alpha < \kappa$ inaccessible $\dot{Q}_\alpha$ is a name for a product of $\text{Sacks}(\alpha, \alpha^{++}) \times \text{Add}(\alpha^{++}, \alpha^{+4})$, and just $\text{Sacks}(\kappa, \kappa^{++})$ for $\alpha = \kappa$. Let $G * g$ be a $\mathbb{P}_\kappa * \dot{Q}_\kappa$-generic. To lift, we need to find an $M[G]$-generic for $R = \text{Add}(\kappa^{++}, \kappa^{+4})$ of $M[G]$.

Lemma 8 *If F is $V[G]$-generic for $R^V = \text{Add}(\kappa^{++}, (\kappa^{+4})^M)$ and $H(\kappa^{++}) \subseteq M$, then $F \cap R$ is $M[G]$-generic.*

Proof. First note that $M[G]$ contains all bounded subset of κ^{++}, and so $R \subseteq R^V$. We show that every max antichain $A \in M[G]$ in R stays maximal in R^V. In $M[G]$, A has size at most κ^{++}, and so $\text{supp}(A) = \bigcup\{\text{supp}(p) \mid p \in A\}$ has size at most κ^{++}. Let $q \in R^V$ be given. Look at q restricted to $\text{supp}(A)$, and denote this condition q^*. Let $f : \text{supp}(A) \to \kappa^{++}$ be 1-1 with $f \in M[G]$. Then q restricted to $f[\text{supp}(A)]$ is in $H(\kappa^{++})$, and so is in $M[G]$. Also q^* is in $M[G]$ since f is. It follows that q^* must be compatible with an element in A, and hence also q. $\square$

However, if a single subset of κ^+ is missing in $M[G]$, this fails.

Lemma 9 *Assume there is $X \subseteq \kappa^+$ which is in $V[G]$ but not in $M[G]$. Then $F \cap R$ is never generic.*

Proof. Assume for simplicity that R is just $\mathsf{Add}(\kappa^{++}, 1)$ of $M[G]$, and $F \subseteq R^V = \mathsf{Add}(\kappa^{++}, 1)$. It is dense in $V[G]$ that X will appear as a segment in some $p \in F$. It follows that $F \cap R$ is a bounded subset of κ^{++}, and hence never generic. $\square$

Note that if we could force an R-generic h over $V[G * g]$ without collapsing, we could lift to h easily since R is κ^+-distributive in $V[G * g]$. And so κ would remain a measurable cardinal in $V[G * g * h]$ failing GCH.

Q: Does forcing with R collapse cardinals?