An example of a capacity for which all positive Borel sets are thick

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Definition of Choquet's $\mathcal{E}$-capacity on E

Let $\mathcal{E} \subseteq P(E)$ be any lattice , $\emptyset \in \mathcal{E}$. Choquet's $\mathcal{E}$-capacity on E is any function $c : P(E) \to [-\infty, \infty]$ such that:

i. $A \subseteq B \subseteq E$ implies $c(A) \leq c(B)$;

ii. if $A_1 \subseteq A_2 \subseteq \ldots$ is any ascending sequence of subsets of E, then $\lim_{n\to\infty} c(A_n) = c(\bigcup_{n=1}^{\infty} A_n)$;

iii. if $E_1 \supseteq E_2 \supseteq \ldots$ is any descending sequence of subsets from $\mathcal{E}$, then $\lim_{n\to\infty} c(E_n) = c(\bigcap_{n=1}^{\infty} E_n)$.

Choquet's capacitability theorem

For any set B from the σ-δ-lattice $\hat{\mathcal{E}}$ (a family closed under countable unions and countable intersections) generated by the family $\mathcal{E}$

the capacity of B can be approximated from below by capacities of subsets of B which are elements of $\mathcal{E}$.

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Definition of a thick set

$B \subseteq \hat{\mathcal{E}}$ is called thick (with respect to a capacity c) if it contains uncountably many pairwise disjoint elements from $\hat{\mathcal{E}}$ of positive capacity c

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- $X = [0, 1] \times \omega_1, \quad c : P(X) \to \{0, 1\},$

  $c(A) = 1 \leftrightarrow \{\alpha < \omega_1 : A \cap [0, 1] \times \{\alpha\} \neq \emptyset\}$ is uncountable

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We construct an example of a $\mathcal{K}(X)$-capacity such that all Borel sets of positive capacity are thick.

Construction of a capacity

μ - the standard probabilistic product measure on X

$\Lambda_1, \Lambda_2, \ldots$ - a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$.

$\Lambda_i = \{n_{i,j} : j \in \mathbb{N}\}$, where $j < k$ implies $n_{i,j} < n_{i,j}$.

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We will now define a family of perfect subsets of X,
$\{C(\xi^{(1)}, \dots, \xi^{(n)}) : \xi^{(1)}, \dots, \xi^{(n)} \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N}\}$.
Let $C(\emptyset) = X$. For $\xi^{(1)}, \dots, \xi^{(n)} \in \{0,1\}^{\mathbb{N}}$ we define
$C(\xi^{(1)}, \dots, \xi^{(n)})$ as

$$C(\xi^{(1)}, \dots, \xi^{(n)}) = \prod_{k \in \mathbb{N}} D_k,$$

where $D_k = \{\xi^{(i)}_j\}$ if $k = n_{i,j} \in \Lambda_i$, $i \le n$, and $D_k = \{0,1\}$ if
$k \notin \bigcup_{i \le n} \Lambda_i$.
Let

$$\nu_{\xi^{(1)}, \dots, \xi^{(n)}} = \prod_{k \notin \bigcup_{i \le n} \Lambda_i} \eta.$$

Let $A \subseteq C(\xi^{(1)}, \ldots, \xi^{(n)})$. Then A is of the form

$$A = \prod_{i \leq n, j \in \mathbb{N}} \{\xi_j^{(i)}\} \times \pi_{X_n}(A),$$

where

$$X_n = \prod_{k \notin \bigcup_{i \leq n} \Lambda_i} \{0, 1\} \quad \text{and} \quad \pi_{X_n} : X \to X_n \text{ is a projection.}$$

Let $A \subseteq C(\xi^{(1)}, \ldots, \xi^{(n)})$ be a Borel set. Let

$$\mu_{\xi^{(1)}, \ldots, \xi^{(n)}}(A) = \nu_{\xi^{(1)}, \ldots, \xi^{(n)}}(\pi_{X_n}(A)).$$

$\mu_{\xi^{(1)},\dots,\xi^{(n)}}$ is a Borel measure on $C(\xi^{(1)},\dots,\xi^{(n)})$. For $A \subseteq X$ let

$$c(A) = \sup\left\{\frac{1}{n}\mu^*_{\xi^{(1)},\dots,\xi^{(n)}}(A \cap C(\xi^{(1)},\dots,\xi^{(n)})) : \xi^{(1)},\dots,\xi^{(n)} \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N}\right\}.$$

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Main Theorem

The function $c : P(X) \to [0,1]$ is non-negative Choquet's $\mathcal{K}(X)$-capacity and if $c(B) > 0$, for a Borel subset B of X, then B contains continuuum many pairwise disjoint Borel subsets of positive capacity.

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- If $\lim_{n\to\infty} c(K_n) = 0$ we have $c(\bigcap_{n=1}^{\infty} K_n) = \lim_{n\to\infty} c(K_n)$.
- If $\lim_{n\to\infty} c(K_n) > 0$, then there exists $m \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{m} \mu_{\xi^{(1,n)},\ldots,\xi^{(m,n)}}(K_n \cap C(\xi^{(1,n)},\ldots,\xi^{(m,n)})) = \lim_{n\to\infty} c(K_n),$$

for some $\xi^{(1,n)},\ldots,\xi^{(m,n)} \in \{0,1\}^{\mathbb{N}}$.

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- For $A_1 \subseteq A_2 \subseteq \dots$ any subsets of X

$$\lim_{n\to\infty} c(A_n) = c\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Proof. Borel positive sets are thick.

Now let B be any Borel subset of X such that $c(B) > 0$. Hence

$$\mu_{\xi^{(1)},\ldots,\xi^{(n)}}(B \cap C(\xi^{(1)},\ldots,\xi^{(n)})) =$$

$$= \nu_{\xi^{(1)},\ldots,\xi^{(n)}}(\pi_{X_n}(B \cap C(\xi^{(1)},\ldots,\xi^{(n)})) > 0,$$

where

$$X_n = \prod_{k \notin \bigcup_{i \le n} \Lambda_i} \{0,1\}^{\mathbb{N}}.$$

for some $\xi^{(1)},\ldots,\xi^{(n)} \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. We have

$$\nu_{\xi^{(1)},\ldots,\xi^{(n)}} = \nu_{\xi^{(1)},\ldots,\xi^{(n)}} \times \prod_{k \in \Lambda_{n+1}} \eta.$$

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$$\mu_{\xi^{(1)},\ldots,\xi^{(n)},\xi}(B \cap C(\xi^{(1)}, \ldots, \xi^{(n)}, \xi))) =$$

$$= \int_{\prod_{k\in\Lambda_{n+1}}\{0,1\}^{\mathbb{N}}} \nu_{\xi^{(1)},\ldots,\xi^{(n)},\xi}(\pi_{X_{n+1}}(B\cap C(\xi^{(1)}, \ldots, \xi^{(n)}, \xi)))d\left(\prod_{k\in\Lambda_{n+1}} \eta\right)(\xi).$$

Thus the set of those $\xi \in \prod_{k\in\Lambda_{n+1}}\{0,1\}$ for which

$$\nu_{\xi^{(1)},\ldots,\xi^{(n)},\xi}(\pi_{X_{n+1}}(B \cap C(\xi^{(1)}, \ldots, \xi^{(n)}, \xi))) > 0$$

has positive measure and hence also

$$c(B \cap C(\xi^{(1)}, \ldots, \xi^{(n)}, \xi)) > 0,$$

must have positive measure and thus it must have cardinality $\mathfrak{c}$.

□

Thank You for Your Attention!