Reflection numbers under large continuum

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http://fuchino.ddo.jp/index-j.html

Winter School in Abstract Analysis 2016
section Set Theory and Topology

(6. Februar 2016 (13:03 JST) version)

5. February 2016, (於 Hejnice)

This presentation is typeset by pLaTeX with beamer class.
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Reflection numbers

- For a class $\mathcal{C}$ of structures with a notion $\sqsubseteq_{\mathcal{C}}$ of substructures, $A \in \mathcal{C}$ and a cardinal $\kappa \leq |A|$, let

$$S^{\mathcal{C}}_{<\kappa}(A) = \{B \in \mathcal{C} : B \sqsubseteq_{\mathcal{C}} A, |B| < \kappa\}.$$

  We identify elements of $\mathcal{C}$ with their underlying sets and consider $S^{\mathcal{C}}_{<\kappa}(A) \subseteq [A]^{<\kappa}$.

- We assume that $S^{\mathcal{C}}_{<\kappa}(A)$ contains a club $\subseteq [A]^{<\kappa}$ for all $A \in \mathcal{C}$.

- For $\mathcal{C}$ as above and a property P, the **reflection number of P in $\mathcal{C}$** is defined as:

$$\mathfrak{Refl}(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} : \text{for all } A \in \mathcal{C} \text{ if } A \not\models P \text{ then} \\ \qquad\qquad\qquad\qquad \text{there is club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \\ \qquad\qquad\qquad\qquad \text{with } B \not\models P\}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } \{\kappa \in \mathrm{Card} : \cdots\} \neq \emptyset; \\ \infty, \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Reflection numbers (2/2)

- **If the property P is hereditary**, i.e. if $A \sqsubseteq_{\mathcal{C}} B$ and $B \models P$ always impies $A \models P$ then the reflection number can be more simply represented as:

$$\mathfrak{Refl}(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} : \text{for all } A \in \mathcal{C} \text{ if } A \not\models P \text{ then} \\ \qquad \text{there is } B \in S^{\mathcal{C}}_{<\kappa}(A) \\ \qquad \text{with } B \not\models P\}, & \text{if } \{\kappa \in \mathrm{Card} : \cdots\} \neq \emptyset\,; \\ \infty, & \text{otherwise.} \end{cases}$$

Examples of reflection numbers

- For $\mathcal{C} =$ trees and $P \Leftrightarrow$ being special
  - The assertion $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ is known as **Rado's Conjecture**.
  - We shall denote this reflection number with $\mathfrak{Refl}_{\text{Rado}}$.
- $\aleph_1 < \mathfrak{Refl}_{\text{Rado}} \leq \infty$. $V = L$ ($\square_\kappa$ for class many κ) $\Rightarrow \mathfrak{Refl}_{\text{Rado}} = \infty$.
  - $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ can be forced starting from a strongly compact cardinal (Todorcěvić 1983).
  - $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies strong forms of Chang's Conjecture (Todorčević 1993, Doebler 2013, S.F.-Sakai-Torres-Usuba ∞).
  - $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević 1993).
  - $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies the Fodor-type Reflection Principle (FRP) and hence all consequences of FRP like SCH (S.F.-Rinot 2011), stationarity reflection (of sets of ordinals of countable cofinality) etc. (S.F.-Sakai-Torres-Usuba ∞).

Examples of reflection numbers (2/3)

- For $\mathcal{C} =$ partial orderings and
  $P \Leftrightarrow$ union of countably many chains (w.r.t. the partial ordering)
  - The assertion $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ for these $\mathcal{C}$ and P is known as **Galvin's Conjecture**.
  - It is still **open** if Galvin's Conjecture is consistent.
  - We shall denote this reflection number with $\mathfrak{Refl}_{\mathsf{Galvin}}$.
- For $\mathcal{C} =$ graphs and
  $P \Leftrightarrow$ of countable chromatic number
  - We shall denote the reflection number $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P with $\mathfrak{Refl}_{chr}$
- $\beth_\omega < \mathfrak{Refl}_{chr}$ (Erdős and Hajnal 1966).
- We have

$$\mathfrak{Refl}_{\mathsf{Rado}} \leq \mathfrak{Refl}_{\mathsf{Galvin}} \leq \mathfrak{Refl}_{chr} \leq \omega_1\text{-strongly compact cardinal.}$$

Examples of reflection numbers (3/3)

$\mathfrak{Refl}_{\text{Rado}} \leq \mathfrak{Refl}_{\text{Galvin}} \leq \mathfrak{Refl}_{chr} \leq \omega_1$-strongly compact cardinal.

- κ is called the **ω_1-strongly compact cardinal** if it is the smallest cardinal κ with the property that for any $\mathcal{L}_{\omega_1,\omega}$ theory T, whenever all subtheories of T of size $< \kappa$ are satisfiable ($< \kappa$-satisfiable) then T itself is also satisfiable.
- For $\mathcal{C} =$ Boolean algebras and $P \;\Leftrightarrow\;$ free
  - We denote the reflection number $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P by $\mathfrak{Refl}^{Ba}_{\text{free}}$. Similarly $\mathfrak{Refl}^{gp}_{\text{free}}$ and $\mathfrak{Refl}^{agp}_{\text{free}}$ for groups and abelian groups.
  - $\aleph_1 < \mathfrak{Refl}^{Ba}_{\text{free}},\ \mathfrak{Refl}^{gp}_{\text{free}},\ \mathfrak{Refl}^{agp}_{\text{free}} \leq \infty$
  - **Open.** Can $\mathfrak{Refl}^{Ba}_{\text{free}},\ \mathfrak{Refl}^{gp}_{\text{free}},\ \mathfrak{Refl}^{agp}_{\text{free}}$ be different?
  - $\mathfrak{Refl}^{gp}_{\text{free}},\ \mathfrak{Refl}^{agp}_{\text{free}} \leq \omega_1$-strongly compact cardinal.
  - **Open?** $\mathfrak{Refl}^{Ba}_{\text{free}} \leq \omega_1$-strongly compact cardinal ?

Large Continuum

- $2^{\aleph_0}$ can be consistently "large" in the following sense:

$2^{\aleph_0}$ is weakly inaccessible.

There are stationarily many weakly inaccessible below $2^{\aleph_0}$.

There is an inner model M with the same cardinals s.t. $2^{\aleph_0}$ is a large cardinal in M.

etc. etc.

Sketch of a consistency proof of Rado's Conjecture

- Suppose that κ is strongly compact and $\mathbb{P} = \mathrm{Col}(\kappa, \omega_1)$. We show that $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$".
- Let G be $(\mathbb{P}, V)$-generic and $T \in V[G]$ a tree s.t.
  (*) $V[G] \models \forall T' \in [T]^{<\aleph_2}$ is special. Note that $(\aleph_2)^{V[G]} = \kappa$.
- ▷ We have to show: $V[G] \models T$ is special.
- In $V[G]$, let $\lambda = |\,T\,|$. Let $j : V \overset{\prec}{\to} M$ be the strongly compact embedding with $j(\kappa) > \lambda$. Let $\mathbb{P}^* = j(\mathbb{P})$ and let G^* be a $(\mathbb{P}^*, V)$-generic set with $G \subseteq G^*$.
- ▷ Let $j^* : V[G] \overset{\prec}{\to} M[G^*]$; $[\underset{\sim}{a}]^G \mapsto [j(\underset{\sim}{a})]^{G^*}$. Let $T^* = j^*(T)$ and let T' be s.t. $j^{*\,\prime\prime}T \subseteq T'$ and $T' \in [T^*]^{\aleph_1} \cap M[G^*]$. Thus $M[G^*] \models T' \in [T^*]^{<\aleph_2}$. By elementarity of j^* and (*), $M[G^*] \models T'$ is special. Hence $V[G^*] \models j''T \cong T$ is special.
- By the following Lemma, T is special even in $V[G]$:

Lemma 1 (Todorčević 1983). For any tree T and σ-closed p.o. $\mathbb{Q}$ if $\Vdash_{\mathbb{Q}}$ "T is special" then T is special.

Cohen forcing

Lemma 1 (Todorčević 1983). For any tree T and σ-closed p.o. $\mathbb{Q}$ if $\Vdash_{\mathbb{Q}}$ " T is special" then T is special.

Proposition 2. For any tree T and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ for any κ if $\Vdash_{\mathbb{P}}$ " T is special" then T is special.

Proof.

- If $\kappa \leq 2^{\aleph_0}$ then $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ is σ-centered and $\Vdash_{\mathbb{P}}$ " T is special" clearly implies that T is special.
- For $\kappa > 2^{\aleph_0}$, suppose that $\Vdash_{\mathbb{P}}$ " T is special". Let $\mathbb{Q} = \mathrm{Col}(\kappa^+, \omega_1)$.

  We have $\Vdash_{\mathbb{P} * \underset{\sim}{Q}}$ " T is special" where $\underset{\sim}{Q}$ is s.t. $\mathbb{Q} * \mathbb{P} \cong \mathbb{P} * \underset{\sim}{Q}$.

  Since $\Vdash_{\mathbb{Q}}$ " $\mathbb{P}$ is σ-centered" it follows that $\Vdash_{\mathbb{Q}}$ " T is special".

  Thus, by Todorcevic's Lemma 1, T is special. □

Adding strongly compact many Cohen reals

Proposition 2. For any tree T and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ for any κ if $\Vdash_{\mathbb{P}}$ " T is special " then T is special.

- Similarly to the consistency proof of Rado's conjecture, Proposition 2 above can be used to prove:

Theorem 3. If κ is a strongly compact cardinal then, letting $\mathbb{P} = \mathrm{Fn}(\lambda, 2)$ for some $\lambda \geq \kappa$, we have $\Vdash_{\mathbb{P}}$ " $\mathfrak{Refl}_{\mathsf{Rado}} = 2^{\aleph_0}$ ". In particular, assertions

"$\mathfrak{Refl}_{\mathsf{Rado}} \leq 2^{\aleph_0}$ + the continuum is very large" and

"$\mathfrak{Refl}_{\mathsf{Rado}} < 2^{\aleph_0}$"

are consistent.

- Remember $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2 \Rightarrow 2^{\aleph_0} \leq \aleph_2$ (Todorcevic).

Question. (M. Viale) $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_3 \Rightarrow 2^{\aleph_0} \leq \aleph_3$?

Indesctructible reflection numbers

- For a class $\mathcal{C}$ of structures and a property P, let us say that $A \in \mathcal{C}$ is **indectructibly** $\neg P$ if $\Vdash_{\mathbb{P}}$ "$A \models \neg P$" for any σ-closed p.o. $\mathbb{P}$.
- For a class $\mathcal{C}$ of structures and a property P, the **indestructible reflection number of P in $\mathcal{C}$** is defined as:

$$\mathfrak{Refl}^*(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} : \text{if } A \in \mathcal{C} \text{ is indestructively } \neg P \text{ then there is club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \text{ with } B \not\models P\}, & \text{if } \{\kappa \in \mathrm{Card} : \cdots\} \neq \emptyset; \\ \infty, & \text{otherwise.} \end{cases}$$

- Let $\mathfrak{Refl}^*_{\mathrm{Galvin}}$ and $\mathfrak{Refl}^*_{chr}$ be $\mathfrak{Refl}^*$ variations of $\mathfrak{Refl}_{\mathrm{Galvin}}$ and $\mathfrak{Refl}_{chr}$.

▷

$$\begin{array}{ccccc} & & \mathfrak{Refl}_{\mathrm{Galvin}} & \leq & \mathfrak{Refl}_{chr} \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \mathfrak{Refl}_{\mathrm{Rado}} & \leq & \mathfrak{Refl}^*_{\mathrm{Galvin}} & \leq & \mathfrak{Refl}^*_{chr}. \end{array}$$

Further consistency results

- $$\begin{array}{ccccc} & & \mathfrak{Refl}_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}_{chr} \\ & & \vee\!| & & \vee\!| \\ \mathfrak{Refl}_{\mathsf{Rado}} & \leq & \mathfrak{Refl}^*_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}^*_{chr}. \end{array}$$

- Arguments similar to that of Theorem 3. amounts to the following theorems:

Theorem 4. For a strongly compact cardinal κ and $\mathbb{P} = \mathrm{Col}(\kappa, \omega_1)$, we have $\Vdash_{\mathbb{P}}$ " $\mathfrak{Refl}^*_{chr} = \aleph_2$ ".

Theorem 5. For a strongly compact cardinal κ and $\mathbb{P} = \mathrm{Fn}(\lambda, 2)$ for $\lambda \geq \kappa$, we have $\Vdash_{\mathbb{P}}$ " $\mathfrak{Refl}^*_{chr} \leq \kappa \leq \lambda = 2^{\aleph_0}$ ".

Theorem 6. For a measurable cardinal κ and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, we have
$\Vdash_{\mathbb{P}}$ " for any graph Γ of size continuum and uncountable chromatic number there exists a subgraph of size $<$ continuum with uncountable chromatic number ".

Martin's Axiom

- Let $\mathfrak{ma}$ be the first number of dense sets for which Martin's Axiom fails. Thus $\omega_1 \leq \mathfrak{ma} \leq 2^{\aleph_0}$.

Theorem 7. (Baumgartner-Malitz-Reihhardt, 1970) Any tree of size $< \mathfrak{ma}$ without uncountable chain is special. □

- Let $T_{\mathbb{R}} = \{t : t \text{ is a strictly increasing sequences in } \mathbb{R} \text{ of successor length} < \omega_1\}$ be considered as a tree with endextension.

Theorem 8. (Todorcevic, 1983) $T_{\mathbb{R}}$ is not special.

Corollary 9. $\mathfrak{ma} < \mathfrak{Refl}_{\text{Rado}}$.

Theorem 10. (Folklore, (S.F., 1992)) If $\mathbb{P}$ has the c.c.c. then, for any A in an universal algebra $\mathcal{C}$, if $\Vdash_{\mathbb{P}}$ "A is free" then A is free.

Corollary 11. If κ is a supercompact cardinal then for the canonical c.c.c. p.o. $\mathbb{P}$ forcing $\kappa = 2^{\aleph_0}$ and $\mathfrak{ma} = 2^{\aleph_0}$ (i.e. MA), $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}^{\mathcal{C}}_{\text{free}} \leq 2^{\aleph_0}$" for any universal algebra $\mathcal{C}$. In particular $\mathfrak{Refl}^{\mathcal{C}}_{\text{free}} < \mathfrak{Refl}_{\text{Rado}}$ is consistent.

Děkuji vám za pozornost !