The tree property at the double successor of a singular cardinal with a larger gap

Radek Honzik
joint with Sy Friedman and Šárka Stejskalová

Department of Logic
Charles University in Prague

logika.ff.cuni.cz/radek

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Basic facts

We list basic facts concerning the tree property.

- ω is *compact* in the sense that for every $n < \omega$, $2^n < \omega$, and every ω-tree has a branch of size ω. In the logical sense, this ensures compactness of the classical logic $L_{\omega,\omega}$.

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- A regular cardinal $\kappa > \omega$ is called *weakly compact* if for every $\mu < \kappa$, $2^\mu < \kappa$, and every κ-tree has a branch of size κ. Existence of such a κ ensures compactness for logic $L_{\kappa,\kappa}$.

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- Let us write TP(κ) (κ has the *tree property*) if every κ-tree has a branch of size κ. Thus TP(ω), and TP(κ) whenever κ is weakly compact.
- A counterexample to TP(κ) is called a *κ-Aronszajn tree*.

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- (Magidor,Shelah). In ZFC, if λ is a singular limit of strongly compact cardinals, then TP(λ^+).
- With GCH, $\neg$TP(κ^{++}) for every $\kappa \geq \omega$.

The tree property can hold at small cardinals, giving them strong compactness-type properties:

- (Mitchell). Suppose κ is weakly compact in V. Then there is a forcing $\mathbb{M}$ such that in $V^{\mathbb{M}}$, ω_1 is preserved, $\kappa = \aleph_2$, and TP($\aleph_2$).

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- Suppose $\mathsf{TP}(\aleph_2)$, then L thinks $\aleph_2$ is weakly compact.
- Successive cardinals with the tree property are harder to get: (Abraham) Suppose $\kappa < \lambda$ are supercompact in V. Then there is a forcing extension $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $\kappa = \aleph_2$, $\lambda = \aleph_3$, and $\mathsf{TP}(\aleph_2)$, $\mathsf{TP}(\aleph_3)$.

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- Current best in this direction: (Neeman). From infinitely many supercompact cardinals, one can get TP at all regular cardinals in the interval $[\aleph_2, \aleph_{\omega+1}]$.

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- (Foreman). If κ is supercompact and $\lambda > \kappa$ is weakly compact, there is a forcing notion $\mathbb{R}$ such that in $V^{\mathbb{R}}$, κ is singular strong limit with cofinality ω, $\lambda = \kappa^{++}$, $2^\kappa = \kappa^{++}$, and TP(κ^{++}).
- (Friedman, Halilovic). If κ is hypermeasurable and $\lambda > \kappa$ is weakly compact, there is a forcing notion $\mathbb{R}$ such that in $V^{\mathbb{R}}$, κ is $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and TP($\aleph_{\omega+2}$).

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Results indicate that it is more likely a restriction of a method.

Theorem (Friedman, H., Stejskalová, 2015)

Suppose κ is supercompact, $\lambda > \kappa$ weakly compact. Then there is a forcing notion $\mathbb{R}$ such that in $V^{\mathbb{R}}$:

1. *κ is singular strong limit with cofinality ω, $\kappa^{++} = \lambda$,*
2. *$2^\kappa = \kappa^{+++}$,*
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Note that κ^{+++} is just an example; any larger cardinal with cofinality larger than κ is possible here.

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In the rest of the talk, we describe the forcing $\mathbb{R}$, starting with Mitchell's and Foreman's forcing.

Mitchell's forcing

Suppose we wish to get TP($\aleph_2$) and GCH. We need to enlarge 2^ω, and collapse κ to $\aleph_2$ while ensuring we kill all κ Aronszajn trees. It requires the correct mix of Knaster and "σ-closed" forcings.

To define $\mathbb{M}$, let $P(\alpha)$ for an ordinal $\alpha > 0$ denote the α-product of the Cohen forcing at ω, and Add(ω_1) be the Cohen forcing at ω_1.

Mitchell's forcing

Definition

A condition in $\mathbb{M}$ is a pair (p, q) such that p is a condition in $P(\kappa)$, and q is a function with at most countable domain $\mathrm{dom}(q) \subseteq \kappa$, such that for all $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is a $P(\alpha)$-name for a condition in $\mathrm{Add}(\omega_1, 1)^{V^{P(\alpha)}}$.

Ordering on the q coordinate is given by what $p|\alpha$ forces in $P(\alpha)$.

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This is important for the application of a large cardinal reflection argument (blackboard).

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Naively, to singularize κ and force $\mathsf{TP}(\kappa^{++})$, we might consider forcing with $\mathbb{M}(\kappa,\lambda) * \mathsf{Prk}_U(\kappa)$. However, it is not clear whether we would not add a κ^{++}-Aronszajn tree in the process, violating $\mathsf{TP}(\kappa^{++})$.

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We need to be more careful.

Let U be a normal measure in $V[G]$, where G is $\mathrm{Add}(\kappa, \lambda)$-generic. For some unbounded set $A \subseteq \kappa$ of inaccessible cardinals, $U \cap V[G|\alpha]$ is a normal measure in $V[G|\alpha]$, $\alpha \in A$. Thus there are projections π_α for $\alpha \in A$:

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$$\pi_\alpha : \mathsf{Add}(\kappa, \lambda) * \mathsf{Prk}_U(\kappa) \to \mathsf{RO}(\mathsf{Add}(\kappa, \alpha) * \mathsf{Prk}_U(\kappa)).$$

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Now define $\mathbb{R}$ as the Mitchell forcing, but with

$\mathsf{Add}(\kappa, \lambda) * \mathsf{Prk}_U(\kappa)$ in place of $\mathsf{Add}(\kappa, \lambda)$, and with the projections π_α, $\alpha \in A$. (Blackboard)

F-H-S, definition of forcing

To ensure $2^\kappa = \kappa^{+3}$ in a version of Foreman's forcing we need to build in the longer Cohen in the Mitchell-style forcing; the naive approach of adding new subsets of κ afterwards does not work because once κ is singular with cofinality ω, adding new subsets of κ tends to collapse κ to ω.

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- There is $\beta \in [\lambda, \lambda^+)$ such that the normal measure U in $V^{\mathrm{Add}(\kappa,\lambda^+)}$ restricts to a measure in $V^{\mathrm{Add}(\kappa,\beta)}$.
- Let π be a bijection between β and $\mathrm{Even}(\lambda)$ (even coordinates of λ). Then the π-image of $U|\beta$ is a normal measure in $V^{\mathrm{Add}(\kappa,\mathrm{Even}(\lambda))}$.

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- There is an unbounded set $B \subseteq \lambda$ of inaccessible cardinals where the π-image of U restricts to a measure in $V^{\mathrm{Add}(\kappa,\mathrm{Even}(\alpha))}$, $\alpha \in B$.

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- There is an unbounded set $B \subseteq \lambda$ of inaccessible cardinals where the π-image of U restricts to a measure in $V^{\mathsf{Add}(\kappa,\mathsf{Even}(\alpha))}$, $\alpha \in B$.
- There are projections $\sigma_\alpha^{\lambda^+}$, $\alpha \in B \subseteq \lambda$:

$$\sigma_\alpha^{\lambda^+} : \mathsf{Add}(\kappa, \lambda^+) * \mathsf{Prk}_U(\kappa) \to \mathsf{RO}(\mathsf{Add}(\kappa, \mathsf{Even}(\alpha)) * \mathsf{Prk}_{\pi(U)}(\kappa)).$$

- Define the forcing $\mathbb{R}$ as in Foreman's forcing, but with the collapsing part only extending to λ (blackboard).

Details

- Why Even(α)? (blackboard)
- Why does it work? (blackboard)

Open questions

1. Can we add collapses to the forcing and have for $\aleph_\omega$ strong limit, $\kappa = \aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+3}$ and $\mathsf{TP}(\aleph_{\omega+2})$?

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