Comparison Game On Trace Ideals

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section Set Theory & Topology
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Definition

Let $X$ be a countable set and $\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{I}$ is called an **ideal** if:

1. $[X]<\omega \subseteq \mathcal{I}$.
2. $B \in \mathcal{I}, A \subseteq B \implies A \in \mathcal{I}$.
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called **$P$-ideal** if

4. $\forall \{B_n : n \in \omega\} \subseteq \mathcal{I}, \exists B \in \mathcal{I}(\forall n \in \omega B_n \subseteq^* B)$.

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Definition (M. Hrušák and D. M. Alcántara)

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\omega$. The **Comparison Game** for $\mathcal{I}$ and $\mathcal{J}$ denoted by $G(\mathcal{I}, \mathcal{J})$ is played as follow:

$$
\begin{array}{c c c c c c c c}
\rule{0pt}{2.5ex} & I_0 & \in & \mathcal{I} & \cdots & I_n & \in & \mathcal{I} & \cdots \\
\hline
II & J_0 & \in & \mathcal{J} & \cdots & J_n & \in & \mathcal{J} & \cdots \\
\end{array}
$$

Player II wins if $\bigcup_{n \in \omega} I_n \in \mathcal{I} \iff \bigcup_{n \in \omega} J_n \in \mathcal{J}$.

We write $\mathcal{I} \subseteq \mathcal{J}$ if Player II has a winning strategy in $G(\mathcal{I}, \mathcal{J})$. And $\mathcal{I} \sim \mathcal{J}$ if $\mathcal{I} \subseteq \mathcal{J} \land \mathcal{J} \subseteq \mathcal{I}$. 

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What is the structure of (Borel ideals/$\simeq$, $\sqsubseteq$)?
What is the structure of $(\text{Borel ideals}/\sim, \sqsubseteq)$?
Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $\omega$.
A function $f: \mathcal{I} \to \mathcal{J}$ is called **Tukey** if

$$\forall A \in \mathcal{J} \exists B \in \mathcal{I} \forall C \in \mathcal{I} (f(C) \subseteq A \Rightarrow C \subseteq B).$$

We write

$$\mathcal{I} \leq_{MT} \mathcal{J}$$

if there is a monotone Tukey function from $\mathcal{I}$ to $\mathcal{J}$.

**Connection:** $\mathcal{I} \leq_{MT} \mathcal{J} \implies \mathcal{I} \subseteq \mathcal{J}$. 
Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $\omega$. A function $f : \mathcal{I} \rightarrow \mathcal{J}$ is called **Tukey** if

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Motivation

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Let $I$ and $J$ be two ideals on $\omega$.

A function $f : I \longrightarrow J$ is called **Tukey** if

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**Connection:** $I \leq_{MT} J \implies I \subseteq J$. 
Definition (W. Wadge)

Let $X, Y \subseteq \omega^\omega$. The **Wadge Game** for $X$ and $Y$ denoted by $W(X, Y)$ is played as follow:

<table>
<thead>
<tr>
<th></th>
<th>$x_0 \in \omega$</th>
<th>$\cdots$</th>
<th>$x_n \in \omega$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$y_0 \in \omega$</td>
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<td>$\cdots$</td>
</tr>
</tbody>
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Denote $x = x_0x_1\ldots x_n\ldots$ and $y = y_0y_1\ldots y_n\ldots$. Player II wins if $x \in X \iff y \in Y$.

We write $X \leq_W Y$ if Player II has a winning strategy in $W(X, Y)$. And $X \equiv_W Y$ if $X \leq_W Y \land Y \leq_W X$. 

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Theorem (M. Hrušák, D. M. Alcántara)

\[ I \sqsubseteq J \iff \tilde{I} \leq_W \tilde{J}, \text{ where } \tilde{I} = \{ x \in \omega^\omega : \text{rang}(x) \in I \}. \]

Corollary

- The game \( G(I, J) \) is determined for every pair \( I, J \) of Borel ideals.
- The order \( \sqsubseteq \) is well-founded.
- The comparison game is almost linear (all antichains have size at most 2).
- There are uncountable many \( \sim \)-classes.
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Theorem (M. Hrušák, D. M. Alcántara)

1. For any Borel ideal $\mathcal{I}$. $\mathcal{I}$ is $F_\sigma$ if and only if $\mathcal{I} \simeq \text{Fin}$.
2. There are at least two classes of $F_{\sigma\delta}(\Pi_3^0)$ non-$F_\sigma(\Sigma_2^0)$-ideals.
3. Let $\mathcal{I}$ be an analytic $P$-ideal. Then $\mathcal{I} \simeq \text{Fin}$ or $\mathcal{I} \simeq \emptyset \times \text{Fin}$.
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Question (M. Hrušák, D. M. Alcántara)

1. Is the order $\sqsubseteq$ linear?
2. Are there exactly two classes of $F_{\sigma\delta}$ non $F_{\sigma}$-ideals?
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**Definition**

Let $X$ be a Borel subset of $2^{\omega}$. The **trace ideal** of $X$, denoted by $T(X)$, is the ideal on $<_{\omega} 2$ generated by $\{\{x | n : n \in \omega\} : x \in X\}$.

**Proposition (van. Engelen 1994)**

Let $\Gamma \supseteq \Delta(D_\omega(\Sigma^0_2))$ be a Wadge degree such that $\forall n \in \omega \forall X \in \Gamma \Rightarrow X^n \in \Gamma$. If $X$ is $\Gamma$, then $T(X)$ is $\Gamma$.

A subset $A \subseteq 2^{\omega}$ is $D_\omega(\Sigma^0_2)$ if there is a increasing $\Sigma^0_2$ sequence $\{B_n : n \in \omega\}$ such that $A = \bigcup_{k\in\omega} B_{2k+1} \setminus B_{2k}$.

**Lemma (van. Engelen)**

If $\mathcal{I}$ is an infinite Borel ideal on $\omega$, then $\mathcal{I} \times \mathcal{I} \equiv_W \mathcal{I}$. 
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Observation: Two $F_{\sigma\delta}$ non-$F_\sigma$-ideals: $\emptyset \times Fin$, $T(Fin^+)$. We have the following result:

**Theorem**

(1). $\emptyset \times Fin \not\subseteq T((\emptyset \times Fin)^+)$. Where
$\emptyset \times Fin = \{ A \subseteq \omega \times \omega : \forall n \in \omega | \{ m : (n, m) \in A \} | < \omega \}.$

(2). $T((\emptyset \times Fin)^+) \not\subseteq \emptyset \times Fin$.

Is the order $\sqsubseteq$ linear?
No
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Theorem

Let $X$ and $Y$ be Borel subset of $\mathcal{P}(\omega)$ with $[X] \supseteq D_2(\Sigma^0_2)$.

$X \leq_W Y \implies T(X) \subseteq T(Y)$. 

Lemma (J. Steel)

Let $\Gamma$ be a Wedge class above $D_2(\Sigma^0_2)$. Then

$\forall A, B ((A \in \Gamma \land B \in \Gamma \setminus \tilde{\Gamma}) \implies A \leq_1 B)$. Where $A \leq_1 B$ means there is injection continuous $f : 2^\omega \rightarrow 2^\omega$ such that $A = f^{-1}(B)$.

The key idea: If the result of Player I plays has finite many anti-chains, use $X \leq_W Y$ control.

If it has infinite many anti-chains, use 1-1 to preserve the result of Player II also has infinite many anti-chains.
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**Theorem**

Let $A \subseteq 2^\omega$ be a Borel subset such that its Wadge class is above $D_\omega(\Sigma^0_2)$ and $B$ be any Borel set. If $B^c \not\leq_W A$, then $T(T(B)^+) \not\subseteq T(A)$.

**Corollary**

If $\mathcal{I}, \mathcal{J}$ be two Borel ideals above $D_\omega(\Sigma^0_2)$, then $\mathcal{I} \equiv_W \mathcal{J} \iff T(\mathcal{I}) \sim T(\mathcal{J})$.

Are there exactly two class of $F_{\sigma\delta}$ non $F_\sigma$-ideals?

How many classes of $F_{\sigma\delta\sigma}$-ideals are there?

No, there are $\omega_1$ many classes.
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Theorem

Let $\mathcal{I}$ be a Borel ideal. Then $T(\mathcal{I}) \subseteq \mathcal{I}$.

But we don’t clear whether $T(\mathcal{I}) \sim \mathcal{I}$,

Form above theorem we have that:

Corollary

For any Borel ideal $\mathcal{I}$ with Wadge class above $D_\omega(\Sigma^0_2)$, we have $T(T(\mathcal{I})) \sim T(\mathcal{I})$. 

Theorem

Let $\mathcal{I}$ be a Borel ideal. Then $T(\mathcal{I}) \subseteq \mathcal{I}$.

But we don’t clear whether $T(\mathcal{I}) \simeq \mathcal{I}$.

Form above theorem we have that:

Corollary

For any Borel ideal $\mathcal{I}$ with Wadge class above $D_\omega(\Sigma^0_2)$, we have $T(T(\mathcal{I})) \simeq T(\mathcal{I})$. 
Definition

For every $0 < \mu < \omega_1$, we let $Fr_{2\mu} = \{S \subseteq \omega^\mu : |S|_L < \omega^\mu\}$,
$Fr_{2\mu+1} = \{S \subseteq \omega^{\mu+1} : \forall m \in \omega((S)_m \in Fr_{2\mu})\}$.

Theorem

For every $0 < \mu < \omega_1$, $T(Fr_{2\mu}) \simeq Fr_{2\mu}$, $T(Fr_{2\mu+1}) \simeq Fr_{2\mu+1}$.

Corollary

Let $\mathcal{I}$ be a Borel ideal. If $Fr_{2\mu} \leq_W \mathcal{I}$, then $Fr_{2\mu} \subseteq \mathcal{I}$ and if $Fr_{2\mu+1} \leq_W \mathcal{I}$, then $Fr_{2\mu+1} \subseteq \mathcal{I}$. 
**Definition**

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Let $\mathcal{I}$ be a Borel ideal. If $Fr_{2\mu} \leq_W \mathcal{I}$, then $Fr_{2\mu} \sqsubseteq \mathcal{I}$ and if $Fr_{2\mu+1} \leq_W \mathcal{I}$, then $Fr_{2\mu+1} \sqsubseteq \mathcal{I}$.
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A dichotomy for analytic $P$-ideal

Theorem (M. Hrušák, D. M. Alcántara)

Let $\mathcal{I}$ be an analytic $P$-ideal. Then $\mathcal{I} \simeq \text{Fin}$ or $\mathcal{I} \simeq \emptyset \times \text{Fin}$.

Theorem

Let $\mathcal{I}$ be an analytic $P$-ideal. If $\mathcal{I}$ is non-$F_\sigma$, then $\mathcal{I}$ is $F_{\sigma_\delta}$-complete.
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Theorem

\[ \emptyset \times \text{Fin} \nsubseteq T((\emptyset \times \text{Fin})^+) \]

The idea of proof.

We construct a winning strategy for Player I in 
\[ G(\emptyset \times \text{Fin}, T((\emptyset \times \text{Fin})^+)) \].

The possible case 1: The maximal anti-chains of result of Player II is finite.

For every \( 1 \leq n < \omega \) we define a game \( G_n \) as follows. In step \( k \), Player I pick a \( I_k \in \emptyset \times \text{Fin} \) and Player II picks a \( J_k \in T((\emptyset \times \text{Fin})^+) \) such that \( \forall i < k \) (\( J_i \subseteq J_k \)) and the maximal cardinal of an antichain of \( J_k \) is \( n \). Player II wins if 
\[ \bigcup_{n \in \omega} I_n \in \emptyset \times \text{Fin} \text{ iff } \bigcup_{n \in \omega} J_n \in T((\emptyset \times \text{Fin})^+) \].

Claim: Player I have a winning strategy \( \sigma_n \) in \( G_n \). (Proved by induction on \( n \))
Theorem

\( \emptyset \times \text{Fin} \not\preceq T((\emptyset \times \text{Fin})^+) \)

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Continue the proof.

The possible case 2: The maximal anti-chains of result of Player II is infinite.

Copy $\emptyset \times Fin\omega$ many times by the structure of $\emptyset \times Fin$.

The strategy of Player I as follows:
Let $\{X_n : n \in \omega\} \subseteq [\omega]^{\omega}$ be a partition of $\omega$. In step 0, Play I plays $\emptyset$, and in step $k > 0$, Let $M(k)$ be the maximal cardinal of an antichain in $\bigcup_{i<k} J_i$.

(1) If $M(k) = M(k-1)$, then Player I plays the game $G_{M(k-1)}$ in $X_{M(k-1)} \times \omega$ think Player II plays $\bigcup_{i<k} J_i$ in a new step and plays follow the winning strategy $\sigma_{M(k-1)}$.

(2) If $M(k) > M(k-1)$, then Player I has to abandon what he has played and begin a new game of $G_{M(k)}$ inside $X_{M(k)} \times \omega$ follow the winning strategy $\sigma_{M(k)}$, and think Player II in step 0 plays $\bigcup_{i<k} J_i$. 
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Děkuji!