κ-strong sequences and the existence of generalized independent families

Joanna Jureczko

Cardinal Stefan Wyszyński University
in Warsaw (Poland)

Winter School in Abstract Analysis
section Set Theory and Topology
Hejnice, 2016

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p \colon p \colon T \to \{0,1\}\}.$$

For $s \subset T$, $i \colon s \to \{0,1\}$ it will be used the following notation

$$H_s^i = \{p \in D^T \colon p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}} \colon \alpha \in T \text{ and } i \colon \{\alpha\} \to \{0,1\}\}$ of the Cantor cube

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p\colon p\colon T \to \{0,1\}\}.$$

For $s \subset T$, $i\colon s \to \{0,1\}$ it will be used the following notation

$$H_s^i = \{p \in D^T : p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}} : \alpha \in T \text{ and } i\colon \{\alpha\} \to \{0,1\}\}$ of the Cantor cube

Definition

A pair (H_s^i, H_v^j) where $|s| < \omega$ is called a connected pair if $H_s^i \cap H_v^j \neq \emptyset$

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p\colon p\colon T \to \{0,1\}\}.$$

For $s \subset T$, $i\colon s \to \{0,1\}$ it will be used the following notation

$$H^i_s = \{p \in D^T : p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}} : \alpha \in T \text{ and } i\colon \{\alpha\} \to \{0,1\}\}$ of the Cantor cube

Definition

A pair (H^i_s, H^j_v) where $|s| < \omega$ is called a connected pair if $H^i_s \cap H^j_v \neq \emptyset$

Definition

A sequence $(H^{i_\alpha}_{s_\alpha}, H^{j_\alpha}_{v_\alpha})$ consisting of connected pairs is called a strong sequence if $H^{i_\alpha}_{s_\alpha} \cap H^{j_\beta}_{v_\beta} = \emptyset$ whenever $\alpha < \beta$.

Theorem (Efimov, 1965)

Let κ be a regular, uncountable cardinal number. In the space D^T a strong sequence

$$(H^{i_\alpha}_{s_\alpha}, H^{j_\alpha}_{v_\alpha}), \alpha < \kappa$$

such that $|s_\alpha| < \omega$ and $|v_\alpha| < \kappa$ for each $\alpha < \kappa$ does not exists.

Let X be a set, and let $\mathscr{B} \subset P(X)$ be a family of non-empty subsets of X closed under finite intersections.
We say that a family $\mathscr{C} \subset \mathscr{B}$ is *centered* iff $\bigcap \mathscr{F} \neq \emptyset$ for each finite subfamily $\mathscr{F} \subset \mathscr{C}$.
Let S be a finite subfamily contained in $\mathscr{B}$ and $H \subseteq \mathscr{B}$. A pair (S, H), will be called *connected* if $S \cup H$ is centered.

Let X be a set, and let $\mathscr{B} \subset P(X)$ be a family of non-empty subsets of X closed under finite intersections.
We say that a family $\mathscr{C} \subset \mathscr{B}$ is *centered* iff $\bigcap \mathscr{F} \neq \emptyset$ for each finite subfamily $\mathscr{F} \subset \mathscr{C}$.
Let S be a finite subfamily contained in $\mathscr{B}$ and $H \subseteq \mathscr{B}$. A pair (S, H), will be called *connected* if $S \cup H$ is centered.

Definition

A sequence (S_ϕ, H_ϕ); $\phi < \alpha$ consisting of connected pairs is called *a strong sequence* if for all λ, in the range $\phi < \lambda < \alpha$, a family $S_\lambda \cup H_\phi$ is not centered.

Theorem (Turzański, 1992)

If for $\mathscr{B} \subset P(X)$ there exists a strong sequence $(S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$, then the family $\mathscr{B}$ contains a subfamily of cardinality λ^+ consisting of pairwise disjoint sets.

Let (X, r) be a set with relation r. (We sometimes write X instead of (X, r) in the situation when it is obvious which relation r is being used).
Let $a, b, c \in X$.
We say that a and b are *compatible* iff there exists c such that

$$(a, c) \in r \text{ and } (b, c) \in r.$$

(We say that a, b have a *bound*).
A set $A \subset X$ is called a κ- *directed* set iff every subset of X of cardinality less than κ has a bound.

Definition

Let (X, r) be a set with a relation r.
A sequence $(H_\phi)_{\phi<\alpha}$, where $H_\phi \subset X$, is called a κ-strong sequence if:
1^o H_ϕ is κ-directed for all $\phi < \alpha$
2^o $H_\psi \cup H_\phi$ is not κ-directed for all $\phi < \psi < \alpha$.

Definition

Let (X, r) be a set with a relation r.
A sequence $(H_\phi)_{\phi<\alpha}$, where $H_\phi \subset X$, is called a κ-strong sequence if:
1^o H_ϕ is κ-directed for all $\phi < \alpha$
2^o $H_\psi \cup H_\phi$ is not κ-directed for all $\phi < \psi < \alpha$.

Theorem (JJ)

Let κ, μ, τ be regular cardinal numbers with $\kappa, \mu < \tau$. If for a set (X, r) of cardinality at least τ there is a κ-strong sequence $\{H_\alpha \subset X : \alpha < \tau\}$ with $|H_\alpha| < \tau$, then there exists a strong sequence $\{T_\alpha : \alpha < \mu\}$ such that $T_\alpha \subset H_\alpha$ and $|T_\alpha| < \kappa$, $\alpha < \mu$.

Theorem (JJ)

Let κ, μ, τ be three cardinals such that $\kappa, \mu < \tau$. Let (X, r), $|X| \geq \tau$ be a set with relation r. Then either X contains a set of cardinality μ which consists of pairwise incompatible elements or X contains a κ-directed subset of cardinality τ.

We need to assume that $\mathscr{A} \subset P(X)$ is *closed under taking* κ *-intersections* i.e. for all $\mathscr{A}' \subset \mathscr{A}$ such that $\mathscr{A}' < \kappa$ we have $\bigcap \mathscr{A}' \in \mathscr{A}$. We introduce below a generalization of a centered family. In the literature, one can find a definition of a σ*-centered family* which says that it is a countable union of centered families. However, we need a different definition of generalized centered family. In order to avoid a confusion, we introduce the following definition.

We need to assume that $\mathscr{A} \subset P(X)$ is *closed under taking* κ *-intersections* i.e. for all $\mathscr{A}' \subset \mathscr{A}$ such that $\mathscr{A}' < \kappa$ we have $\bigcap \mathscr{A}' \in \mathscr{A}$. We introduce below a generalization of a centered family. In the literature, one can find a definition of a σ*-centered family* which says that it is a countable union of centered families. However, we need a different definition of generalized centered family. In order to avoid a confusion, we introduce the following definition.

Definition

Let κ, τ be cardinals with $\kappa < \tau$. A family of sets $\mathscr{A} \subset P(X)$, with $|\mathscr{A}| \geq \tau$, is called a κ*-vaulted* family iff for each subfamily $\mathscr{B} \subset \mathscr{A}$ of cardinality less than κ we have $\bigcap \mathscr{B} \neq \emptyset$.

Theorem (JJ)

Let κ, μ, τ be cardinals with $\kappa, \mu < \tau$. Let $\mathscr{A} \subset P(X)$ be a family of sets with $|\mathscr{A}| \geq \tau$ closed under taking κ - intersections. Then $\mathscr{A}$ contains a subfamily of cardinality μ that consists of pairwise disjoint sets or $\mathscr{A}$ contains a κ-vaulted family of cardinality τ.

Theorem (JJ)

Let κ, μ, τ be cardinals with $\kappa, \mu < \tau$. Let $\mathscr{A} \subset P(X)$ be a family of sets with $|\mathscr{A}| \geq \tau$ closed under taking κ - intersections. Then $\mathscr{A}$ contains a subfamily of cardinality μ that consists of pairwise disjoint sets or $\mathscr{A}$ contains a κ-vaulted family of cardinality τ.

Proof.

Let $\mathscr{A} = \{A_\gamma \colon \gamma < \tau\}$ be a family as it is required in theorem. Define a partial ordered set $\mathscr{P} = \{\gamma < \tau \colon A_\gamma \in \mathscr{A}\}$ with the following relation.

$$(\gamma, \beta) \in r \Leftrightarrow A_\gamma \subset A_\beta.$$

If γ, β are incompatible, then $A_\gamma \cap A_\beta = \emptyset$. According to previous theorem the proof is complete. □

Definition

A family $\{(A^0_\xi, A^1_\xi)\colon \xi < \alpha\}$ of ordered pairs of subsets of X such that $A^0_\xi \cap A^1_\xi = \emptyset$ for $\xi < \alpha$ is called *an independent family* *($\sigma-$independent family)* (of length α) if for each finite (countable) set $F \subset \alpha$ and each function $i : F \to \{0,1\}$ we have that

$$\bigcap\{A^{i(\xi)}_\xi\colon \xi \in F\} \neq \emptyset.$$

Definition

Let $\mathscr{I} = \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ be a family of partitions of infinite set S with each $\lambda_\alpha \geq 2$ and let κ, λ, θ be cardinals. If for any $J \in [\tau]^{<\theta}$ and for any $f \in \Pi_{\alpha \in J} \lambda_\alpha$ the intersection $\bigcap\{I_\alpha^{f(\alpha)} : \alpha \in J\}$ has cardinality at least κ, then $\mathscr{I}$ is called (θ, κ)-generalized independent family on S. Moreover, if $\lambda_\alpha = \lambda$ for all $\alpha < \tau$, then $\mathscr{I}$ is called a $(\theta, \kappa, \lambda)$-generalized independent family on S.

We give below some notions of generalized independent families:.

1. An independent family is $(\omega, 1)-$generalized independent family.

2. A σ-independent family is $(\omega_1, 1)$-generalized independent family.

Theorem (Elser, 2011)

Let $\lambda, \theta, (\lambda \geq \theta)$. On every set with at least $\lambda^{<\theta}$ elements there exists a $(\theta, 1, \lambda)$-generalized independent family of cardinality 2^{λ}.

We denote by $S(X)$ the smallest cardinal κ such that every family of pairwise disjoint nonempty open sets has size less than κ.

Theorem (JJ)

Let κ, τ ($\kappa < \tau$) be cardinals with $\kappa \geq S(X)$. Let $\mathscr{A} \subset P(X)$ be a family of sets ($|\mathscr{A}| \geq \tau$) closed under taking κ-intersections. Then there exists a $(\kappa, 1)$-generalized independent family of cardinality τ.

Definition

Let μ, κ be two cardinals with $\aleph_0 \leq \kappa \leq \mu$ and $\{X_i\}_{i\in\mu}$ be a family of topological spaces. Then $\square^{\kappa}_{i\in\mu} X_i$ denotes the κ-box product which is induced on the full cartesian product $\Pi_{i\in\mu} X_i$ by the canonical base

$$\mathscr{B} = \{\bigcap_{i\in I} pr_i^{-1}(U_i) \colon I \in P_{<\kappa}(\mu) \text{ and } U_i \text{ is open in } X_i\},$$

where $P_{<\kappa}(\mu) := \{I \subset \mu : |I| < \kappa\}$.

Theorem (Hu, 2006)

Let $\mathscr{I}$ be a $(\theta, 1)$-generalized independent family on a set S and let $\{X_\alpha\}_{\alpha<\tau}$ be a family of topological spaces such that $d(X_\alpha) \leq \lambda_\alpha$ for all $\alpha < \tau$. Then $d(\square^{\tau}_{\alpha\in\theta}(X_\alpha)) \leq |S|$.

Corollary (JJ)

Let κ, θ, τ ($\kappa < \tau$) be cardinals with $\kappa \geq S(X)$ and let S be a set. Let $\mathscr{A} \subset P(X)$ be a family of sets ($|\mathscr{A}| \geq \tau$) closed under taking κ-intersections and let $\{X_\alpha\}_{\alpha<\tau}$ be a family of topological spaces such that $d(X_\alpha) \leq \lambda_\alpha$ for all $\alpha < \tau$. Then $d(\square^\tau_{\alpha\in\theta}(X_\alpha)) \leq |S|$.

Definition

Let κ, λ, θ be three cardinals. Let S be an infinite set of the cardinality κ. The cardinal $i(\theta, \kappa, \lambda)$ is the smallest cardinal τ such that there are no $(\theta, 1, \lambda)$- generalized independent families on S of size τ.

We introduce the following invariant

$$\hat{s}_\kappa = \sup\{\alpha : \text{ there exists a } \kappa\text{-strong sequence of size } \alpha\}.$$

Theorem (JJ)

Let κ, λ, θ be three cardinals with $\kappa < \theta$. Let S be a set with $|S| \geq \theta$. Then $\hat{s}_{|S|} \leq i(\theta, |S|, \lambda)$.

Theorem (Hu, 2006)

Let S be a set and let λ, τ, θ be three cardinals with θ infinite. Then the following are equivalent

1) $\tau < i(\theta, |S|, \lambda)$

2) $d(\square_\theta^\tau)(X_\alpha) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha<\tau}$ with each $d(X_\alpha) \leq \lambda$.

Theorem (Hu, 2006)

Let S be a set and let λ, τ, θ be three cardinals with θ infinite. Then the following are equivalent
1) $\tau < i(\theta, |S|, \lambda)$
2) $d(\square_\theta^\tau)(X_\alpha) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha<\tau}$ with each $d(X_\alpha) \leq \lambda$.

Corollary (JJ)

Let S be a set and let λ, θ be three cardinals with θ infinite. Then the following are equivalent
1) $\hat{s}_{|S|} \leq i(\theta, |S|, \lambda)$
2) $d(\square_\theta^{\hat{s}_{|S|}}(X_\alpha)) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha<\hat{s}_{|S|}}$, with each $d(X_\alpha) \leq \lambda$.

The main bibliography

The main bibliography

- J. JURECZKO, M. TURZAŃSKI, 2008 *From a Ramsey-type theorem to independence*, Acta Univ. Car. Math et Phy. **49, 2** (2008), 47-55.

The main bibliography

- J. JURECZKO, M. TURZAŃSKI, 2008 *From a Ramsey-type theorem to independence*, Acta Univ. Car. Math et Phy. **49, 2** (2008), 47-55.
- W. HU, 2006, *Generalized independent families and dense sets of Box-Product spaces*, App. Gen. Top. **7(2)**, (2006), 203-209.

The main bibliography

- J. JURECZKO, M. TURZAŃSKI, 2008 *From a Ramsey-type theorem to independence*, Acta Univ. Car. Math et Phy. **49, 2** (2008), 47-55.
- W. HU, 2006, *Generalized independent families and dense sets of Box-Product spaces*, App. Gen. Top. **7(2)**, (2006), 203-209.
- S. O. ELSER, 2011, *Density of κ-Box Products and the existenxce of generalized independent families*, App. Gen. Top., **12(2)** (2011), 221-225.

The main bibliography

- J. Jureczko, M. Turzański, 2008 *From a Ramsey-type theorem to independence*, Acta Univ. Car. Math et Phy. **49, 2** (2008), 47-55.
- W. Hu, 2006, *Generalized independent families and dense sets of Box-Product spaces*, App. Gen. Top. **7(2)**, (2006), 203-209.
- S. O. Elser, 2011, *Density of κ-Box Products and the existenxce of generalized independent families*, App. Gen. Top., **12(2)** (2011), 221-225.
- J.Jureczko, *κ-strong sequences and the existence of generalized independent families*, preprint.

THANK YOU FOR YOUR ATTENTION