κ-strong sequences and the existence of generalized independent families

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Let $T$ be an infinite set. Denote the Cantor cube by

$$D^T = \{ p : p : T \rightarrow \{0, 1\} \}.$$ 

For $s \subset T$, $i : s \rightarrow \{0, 1\}$ it will be used the following notation

$$H^i_s = \{ p \in D^T : p|s = i \}.$$ 

Efimov defined strong sequences in the subbase

$$\{ H^i_{\alpha} : \alpha \in T \text{ and } i : \{\alpha\} \rightarrow \{0, 1\} \}$$ 

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**Definition**

A pair $(H^i_s, H^j_v)$ where $|s| < \omega$ is called a connected pair if 

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**Definition**

A sequence $(H^i_{s\alpha}, H^j_{v\alpha})$ consisting of connected pairs is called a strong sequence if
\[ H^i_{s\alpha} \cap H^j_{v\beta} = \emptyset \text{ whenever } \alpha < \beta. \]
Theorem (Efimov, 1965)

Let $\kappa$ be a regular, uncountable cardinal number. In the space $D^T$ a strong sequence

$$(H^i_{s\alpha}, H^i_{v\alpha}), \alpha < \kappa$$

such that $|s_\alpha| < \omega$ and $|v_\alpha| < \kappa$ for each $\alpha < \kappa$ does not exists.
Let $X$ be a set, and let $\mathcal{B} \subset P(X)$ be a family of non-empty subsets of $X$ closed under finite intersections. We say that a family $\mathcal{C} \subset \mathcal{B}$ is centered iff $\bigcap F \neq \emptyset$ for each finite subfamily $F \subset \mathcal{C}$.

Let $S$ be a finite subfamily contained in $\mathcal{B}$ and $H \subseteq \mathcal{B}$. A pair $(S, H)$, will be called connected if $S \cup H$ is centered.
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Let $S$ be a finite subfamily contained in $\mathcal{B}$ and $H \subseteq \mathcal{B}$. A pair $(S, H)$, will be called connected if $S \cup H$ is centered.

**Definition**

A sequence $(S_\phi, H_\phi); \phi < \alpha$ consisting of connected pairs is called a strong sequence if for all $\lambda$, in the range $\phi < \lambda < \alpha$, a family $S_\lambda \cup H_\phi$ is not centered.
Theorem (Turzański, 1992)

If for $\mathcal{B} \subset P(X)$ there exists a strong sequence $(S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$, then the family $\mathcal{B}$ contains a subfamily of cardinality $\lambda^+$ consisting of pairwise disjoint sets.
Let \((X, r)\) be a set with relation \(r\). (We sometimes write \(X\) instead of \((X, r)\) in the situation when it is obvious which relation \(r\) is being used).
Let \(a, b, c \in X\).
We say that \(a\) and \(b\) are \emph{compatible} iff there exists \(c\) such that
\[
(a, c) \in r \text{ and } (b, c) \in r.
\]
(We say that \(a, b\) have a \emph{bound}).
A set \(A \subset X\) is called a \(\kappa\)-\emph{directed} set iff every subset of \(X\) of cardinality less than \(\kappa\) has a bound.
Definition

Let \((X, r)\) be a set with a relation \(r\).
A sequence \((H_\phi)_{\phi < \alpha}\), where \(H_\phi \subseteq X\), is called a \(\kappa\)-strong sequence if:
1° \(H_\phi\) is \(\kappa\)-directed for all \(\phi < \alpha\)
2° \(H_\psi \cup H_\phi\) is not \(\kappa\)-directed for all \(\phi < \psi < \alpha\).
Definition

Let $(X, r)$ be a set with a relation $r$. A sequence $(H_\phi)_{\phi < \alpha}$, where $H_\phi \subset X$, is called a $\kappa$-strong sequence if:

1. $H_\phi$ is $\kappa$-directed for all $\phi < \alpha$
2. $H_\psi \cup H_\phi$ is not $\kappa$-directed for all $\phi < \psi < \alpha$.

Theorem (JJ)

Let $\kappa, \mu, \tau$ be regular cardinal numbers with $\kappa, \mu < \tau$. If for a set $(X, r)$ of cardinality at least $\tau$ there is a $\kappa$-strong sequence $\{H_\alpha \subset X : \alpha < \tau\}$ with $|H_\alpha| < \tau$, then there exists a strong sequence $\{T_\alpha : \alpha < \mu\}$ such that $T_\alpha \subset H_\alpha$ and $|T_\alpha| < \kappa$, $\alpha < \mu$. 
Theorem (JJ)

Let $\kappa, \mu, \tau$ be three cardinals such that $\kappa, \mu < \tau$. Let $(X, r)$, $|X| \geq \tau$ be a set with relation $r$. Then either $X$ contains a set of cardinality $\mu$ which consists of pairwise incompatible elements or $X$ contains a $\kappa$-directed subset of cardinality $\tau$. 
We need to assume that $\mathcal{A} \subset P(X)$ is closed under taking $\kappa$-intersections i.e. for all $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{A}' < \kappa$ we have $\bigcap \mathcal{A}' \in \mathcal{A}$. We introduce below a generalization of a centered family. In the literature, one can find a definition of a $\sigma$-centered family which says that it is a countable union of centered families. However, we need a different definition of generalized centered family. In order to avoid a confusion, we introduce the following definition.
We need to assume that $\mathcal{A} \subset P(X)$ is \textit{closed under taking $\kappa$-intersections} i.e. for all $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{A}' < \kappa$ we have $\bigcap \mathcal{A}' \in \mathcal{A}$. We introduce below a generalization of a centered family. In the literature, one can find a definition of a $\sigma$-\textit{centered family} which says that it is a countable union of centered families. However, we need a different definition of generalized centered family. In order to avoid a confusion, we introduce the following definition.

\textbf{Definition}

Let $\kappa, \tau$ be cardinals with $\kappa < \tau$. A family of sets $\mathcal{A} \subset P(X)$, with $|\mathcal{A}| \geq \tau$, is called a $\kappa$-\textit{vaulted} family iff for each subfamily $\mathcal{B} \subset \mathcal{A}$ of cardinality less than $\kappa$ we have $\bigcap \mathcal{B} \neq \emptyset$. 
**Theorem (JJ)**

Let $\kappa, \mu, \tau$ be cardinals with $\kappa, \mu < \tau$. Let $\mathcal{A} \subset P(X)$ be a family of sets with $|\mathcal{A}| \geq \tau$ closed under taking $\kappa$ - intersections. Then $\mathcal{A}$ contains a subfamily of cardinality $\mu$ that consists of pairwise disjoint sets or $\mathcal{A}$ contains a $\kappa$-vaulted family of cardinality $\tau$.
Theorem (JJ)

Let $\kappa, \mu, \tau$ be cardinals with $\kappa, \mu < \tau$. Let $\mathcal{A} \subset P(X)$ be a family of sets with $|\mathcal{A}| \geq \tau$ closed under taking $\kappa$-intersections. Then $\mathcal{A}$ contains a subfamily of cardinality $\mu$ that consists of pairwise disjoint sets or $\mathcal{A}$ contains a $\kappa$-vaulted family of cardinality $\tau$.

Proof.

Let $\mathcal{A} = \{A_\gamma : \gamma < \tau\}$ be a family as it is required in theorem. Define a partial ordered set $\mathcal{P} = \{\gamma < \tau : A_\gamma \in \mathcal{A}\}$ with the following relation.

$$(\gamma, \beta) \in r \iff A_\gamma \subset A_\beta.$$  

If $\gamma, \beta$ are incompatible, then $A_\gamma \cap A_\beta = \emptyset$. According to previous theorem the proof is complete.
A family \( \{ (A^0_\xi, A^1_\xi) : \xi < \alpha \} \) of ordered pairs of subsets of \( X \) such that \( A^0_\xi \cap A^1_\xi = \emptyset \) for \( \xi < \alpha \) is called an independent family (\( \sigma \)-independent family) (of length \( \alpha \)) if for each finite (countable) set \( F \subset \alpha \) and each function \( i : F \to \{0, 1\} \) we have that

\[
\bigcap \{ A^{i(\xi)}_\xi : \xi \in F \} \neq \emptyset.
\]
**Definition**

Let $\mathcal{I} = \{ \{ I^\beta_\alpha : \beta < \lambda_\alpha \} : \alpha < \tau \}$ be a family of partitions of infinite set $S$ with each $\lambda_\alpha \geq 2$ and let $\kappa, \lambda, \theta$ be cardinals. If for any $J \in [\tau]^{<\theta}$ and for any $f \in \prod_{\alpha \in J} \lambda_\alpha$ the intersection $\bigcap\{ I^{f(\alpha)}_\alpha : \alpha \in J \}$ has cardinality at least $\kappa$, then $\mathcal{I}$ is called $(\theta, \kappa)$-generalized independent family on $S$. Moreover, if $\lambda_\alpha = \lambda$ for all $\alpha < \tau$, then $\mathcal{I}$ is called a $(\theta, \kappa, \lambda)$-generalized independent family on $S$. 
We give below some notions of generalized independent families:

1. An independent family is \((\omega, 1)\)-generalized independent family.
2. A \(\sigma\)-independent family is \((\omega_1, 1)\)-generalized independent family.
Theorem (Elser, 2011)

Let $\lambda, \theta, (\lambda \geq \theta)$. On every set with at least $\lambda^\theta$ elements there exists a $(\theta, 1, \lambda)$-generalized independent family of cardinality $2^\lambda$. 
We denote by $S(X)$ the smallest cardinal $\kappa$ such that every family of pairwise disjoint nonempty open sets has size less than $\kappa$.

**Theorem (JJ)**

Let $\kappa, \tau$ ($\kappa < \tau$) be cardinals with $\kappa \geq S(X)$. Let $\mathcal{A} \subset P(X)$ be a family of sets ($|\mathcal{A}| \geq \tau$) closed under taking $\kappa$-intersections. Then there exists a $(\kappa, 1)$-generalized independent family of cardinality $\tau$. 

Definition

Let $\mu, \kappa$ be two cardinals with $\kappa_0 \leq \kappa \leq \mu$ and $\{X_i\}_{i \in \mu}$ be a family of topological spaces. Then $\square_{i \in \mu}^\kappa X_i$ denotes the $\kappa$-box product which is induced on the full cartesian product $\prod_{i \in \mu} X_i$ by the canonical base

$$\mathcal{B} = \left\{ \bigcap_{i \in I} pr_i^{-1}(U_i) : I \in P_{<\kappa}(\mu) \text{ and } U_i \text{ is open in } X_i \right\},$$

where $P_{<\kappa}(\mu) := \{I \subset \mu : |I| < \kappa\}$.

Theorem (Hu, 2006)

Let $\mathcal{I}$ be a $(\theta, 1)$-generalized independent family on a set $S$ and let $\{X_\alpha\}_{\alpha < \tau}$ be a family of topological spaces such that $d(X_\alpha) \leq \lambda_\alpha$ for all $\alpha < \tau$. Then $d(\square_{\alpha \in \theta}^\tau X_\alpha) \leq |S|$. 

Corollary (JJ)

Let \( \kappa, \theta, \tau \ (\kappa < \tau) \) be cardinals with \( \kappa \geq S(X) \) and let \( S \) be a set. Let \( \mathcal{A} \subset P(X) \) be a family of sets (\( |\mathcal{A}| \geq \tau \)) closed under taking \( \kappa \)-intersections and let \( \{X_\alpha\}_{\alpha < \tau} \) be a family of topological spaces such that \( d(X_\alpha) \leq \lambda_\alpha \) for all \( \alpha < \tau \). Then \( d(\bigcap_{\alpha \in \theta}(X_\alpha)) \leq |S| \).
Definition

Let $\kappa, \lambda, \theta$ be three cardinals. Let $S$ be an infinite set of the cardinality $\kappa$. The cardinal $i(\theta, \kappa, \lambda)$ is the smallest cardinal $\tau$ such that there are no $(\theta, 1, \lambda)$-generalized independent families on $S$ of size $\tau$.

We introduce the following invariant

$$\hat{S}_\kappa = \sup\{\alpha : \text{there exists a } \kappa\text{-strong sequence of size } \alpha\}.$$

Theorem (JJ)

Let $\kappa, \lambda, \theta$ be three cardinals with $\kappa < \theta$. Let $S$ be a set with $|S| \geq \theta$. Then $\hat{S}_{|S|} \leq i(\theta, |S|, \lambda)$. 
Theorem (Hu, 2006)

Let $S$ be a set and let $\lambda, \tau, \theta$ be three cardinals with $\theta$ infinite. Then the following are equivalent

1) $\tau < i(\theta, |S|, \lambda)$

2) $d(\bigotimes_{\tau}^{\theta}(X_\alpha)) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha < \tau}$ with each $d(X_\alpha) \leq \lambda$. 

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Let $S$ be a set and let $\lambda, \tau, \theta$ be three cardinals with $\theta$ infinite. Then the following are equivalent

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2) $d(\square_\theta \overrightarrow{s}(X_\alpha)) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha < \tau}$ with each $d(X_\alpha) \leq \lambda$.

Corollary (JJ)

Let $S$ be a set and let $\lambda, \theta$ be three cardinals with $\theta$ infinite. Then the following are equivalent

1) $\hat{s}_{|S|} \leq i(\theta, |S|, \lambda)$

2) $d(\square_\theta \hat{s}_{|S|}(X_\alpha)) \leq |S|$ holds for any family of topological spaces $\{X_\alpha\}_{\alpha < \hat{s}_{|S|}}$, with each $d(X_\alpha) \leq \lambda$. 
The main bibliography


J. Jureczko, $\kappa$-strong sequences and the existence of generalized independent families, preprint.
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