VLADIMIR KANOVEI, Some applications of finite-support products of Jensen’s minimal $\Delta^1_3$ forcing.

IIITP, Moscow, Russia, and MIIT, Moscow, Russia.

E-mail: kanovei@rambler.ru.

Jensen [4] introduced a forcing notion $P \in L$ such that any $P$-generic real $a$ over $L$ has minimal $L$-degree, is $\Delta^1_3$ in $L[a]$, and is the only $P$-generic real in $L[a]$. Further applications of this forcing include iterations, finite products and finite-support infinite products for symmetric choiceless models [1], et cetera. We present some new applications of finite-support infinite products of Jensen’s forcing and its variations.

**Theorem 1** ([5]). There is a generic extension $L[x]$ of $L$ by a real $x$ in which $[x]_{E_0}$ is a (lightface) $\Pi^1_3$ set containing no OD (ordinal-definable) reals. Therefore it is consistent with $\text{ZFC}$ that there is a countable non-empty lightface $\Pi^1_3$ set of reals, in fact a $E_0$ equivalence class, containing no OD elements.

Recall that $E_0$ is an equivalence relation on $\omega^\omega$ such that $x \in E_0 y$ iff $x(k) = y(k)$ for all but finite $k$, and $[x]_{E_0} = \{ y \in \omega^\omega : x \in E_0 y \}$ is the (countable) $E_0$-class of a real $x \in \omega^\omega$.

Let a Groszek – Laver pair be any OD pair of sets $X,Y \subseteq \omega^\omega$ such that neither of $X,Y$ is separately OD. As demonstrated in [3], if $(x,y)$ is a Sacks\times Sacks generic pair of reals over $L$ then their $L$-degrees $X = [x]_L \cap \omega^\omega$ and $Y = [y]_L \cap \omega^\omega$ form such a pair in $L[x,y]$; the sets $X,Y$ is this example are obviously uncountable.

**Theorem 2** ([2]). There is a generic extension $L[a,b]$ of $L$ by reals $a,b$ in which it is true that the countable sets $[a]_{E_0}$ and $[b]_{E_0}$ form a Groszek – Laver pair, and moreover the union $[a]_{E_0} \cup [b]_{E_0}$ is a $\Pi^1_3$ set.

**Theorem 3** ([6]). It is consistent with $\text{ZFC}$ that there is a $\Pi^1_3$ set $\emptyset \neq Q \subseteq \omega^\omega \times \omega^\omega$ with countable cross-sections $Q_x = \{ y : (x,y) \in Q \}, x \in \omega^\omega$, non-uniformizable by any ROD set. In fact each cross-section $Q_x$ in the example is an $E_0$ class.

ROD = real-ordinal-definable. Typical examples of non-ROD-uniformizable sets, like $\{ (x,y) : y \notin L[x] \}$ in the Solovay model, definitely have uncountable cross-sections.

Let analytically definable mean the union $\bigcup_n \Sigma^1_n$ of all lightface definability classes $\Sigma^1_n$. The full basis theorem is the claim that any non-empty analytically definable set $X \subseteq \omega^\omega$ contains an analytically definable element. This is true assuming $V = L$, and generally assuming that there is an analytically definable wellordering of the reals. We prove that the implication is irreversible.

**Theorem 4** (with A. Enayat). It is consistent with $\text{ZFC}$ that the full basis theorem is true but there is no analytically definable wellordering of the reals.


