The complexity of embeddability between groups

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Borel reducibility

In the framework of Borel reducibility, relations are defined over Polish or standard Borel spaces.

Definition

Let E and F be binary relations over X and Y, respectively.

- E **Borel reduces** to F (or $\boldsymbol{E \leq_B F}$) if and only if there is a Borel $f : X \to Y$ such that
$$x_1 \, E \, x_2 \quad \Leftrightarrow \quad f(x_1) \, F \, f(x_2).$$
- E and F are **Borel bi-reducible** (or $\boldsymbol{E \sim_B F}$) if and only if $E \leq_B F$ and $F \leq_B E$.

Compare equivalence relations

The ordering $\leq_B$ can be used to compare equivalence relations.

Examples

(Gromov) the isometry between compact Polish metric spaces Borel reduces to $=_{\mathbb{R}}$.

(Stone) the homeomorphism between compact zero-dimensional Hausdorff spaces Borel reduces to the isomorphism between Boolean algebras.

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Analytic relations

Definition

A relation E defined on X is $\boldsymbol{\Sigma}^1_1$ (or **analytic**) if it is analytic as a subset of $X \times X$.

Examples

- Fix $\mathcal{L}$ a countable relational language. Any countable $\mathcal{L}$-structure is viewed as an element of $X_{\mathcal{L}} = \prod_{R \in \mathcal{L}} 2^{\mathbb{N}^{a(R)}}$

$$M \sqsubseteq_{\mathcal{L}} N \quad \overset{def}{\Longleftrightarrow} \quad \exists h : M \longrightarrow N \quad \text{embedding.}$$

- If X is a Polish space and G is a Polish group such that $a : G \curvearrowright X$ is a Borel action,

$$x \, E^X_G \, y \quad \overset{def}{\Longleftrightarrow} \quad \exists g \text{ such that } a(g,x) = y.$$

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$\boldsymbol{\Sigma}^1_1$ completeness

Definition

An equivalence relation E is **$\boldsymbol{\Sigma}^1_1$-complete** if and only if $F \leq_B E$, for every $\boldsymbol{\Sigma}^1_1$ equivalence relation F.

Definition

A quasi-order Q is **$\boldsymbol{\Sigma}^1_1$-complete** if and only if $P \leq_B Q$, for every $\boldsymbol{\Sigma}^1_1$ quasi-order P.

Example

- isometry between separable Banach spaces, (Ferenczi-Louveau-Rosendal 2009)
- $\cong_{\mathfrak{G}}$ the topological isomorphism between Polish groups. (Ferenczi-Louveau-Rosendal 2009)

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Σ^1_1-complete quasi-orders

Examples

- $\sqsubseteq_{\mathsf{Gr}}$ the embeddability on countable graphs, (Louveau-Rosendal 2005)
- $\sqsubseteq^C$ the continuous embeddability on compact metrizable spaces, (Louveau-Rosendal 2005)
- $\sqsubseteq^i$ the isometric embeddability on separable Banach spaces, (Ferenczi-Louveau-Rosendal 2009)
- $\sqsubseteq_{\mathfrak{G}}$ the topological embeddability on Polish groups, (Ferenczi-Louveau-Rosendal 2009)
- $\sqsubseteq_{\mathsf{Gp}}$ the embeddability on countable groups. (Williams 2014)

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Invariant Universality

Definition

Let S be a $\mathbf{\Sigma}^1_1$ quasi-order and E a $\mathbf{\Sigma}^1_1$ equivalence subrelation of S. We say that the pair (S, E) is **invariantly universal** (or **universal**) if for every $\mathbf{\Sigma}^1_1$ quasi-order R there is a Borel $B \subseteq dom(S)$ such that:

- B is invariant respect to E,
- $S \restriction B \sim_B R$.

(Q, E) invariantly universal $\Rightarrow$ Q è $\mathbf{\Sigma}^1_1$-complete.

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$$(Q, E) \text{ invariantly universal} \quad \Rightarrow \quad Q \text{ è } \mathbf{\Sigma}^1_1\text{-complete.}$$

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Embeddability of countable groups

Theorem (Williams 2014)

$\sqsubseteq_{\mathsf{Gp}}$ *is* $\boldsymbol{\Sigma}^1_1$*-complete.*

Theorem (C.-Motto Ros)

$\sqsubseteq_{\mathsf{Gp}}$ *is invariantly universal.*

The only known technique

There exists a Borel $\mathbb{G} \subseteq X_{\mathsf{Gr}}$ such that $\sqsubseteq_{Gr} \restriction \mathbb{G}$ is $\boldsymbol{\Sigma}^1_1$-complete and over $\mathbb{G}$ equality and isomorphism coincide.

Theorem (Camerlo-Marcone-Motto Ros 2013)

Let S be a $\boldsymbol{\Sigma}^1_1$ quasi-order on X and $E \subseteq S$ a $\boldsymbol{\Sigma}^1_1$ equivalence relation. Assume that there is a Borel $f : \mathbb{G} \to X$ such that:

- $\sqsubseteq_{\mathbb{G}} \leq_B S$ *via* f,
- $=_{\mathbb{G}} \leq_B E$ *via* f,
- *there exists a standard Borel space Y and a Borel reduction g of E to E^Y_H, for some Polish group $H \curvearrowright Y$, such that*

$$\Sigma : \mathbb{G} \longrightarrow F(H)$$
$$T \longmapsto \{h \in H : h \cdot (g \circ f(T)) = g \circ f(T)\} \quad \text{is Borel.}$$

Then, (S, E) is invariantly universal.

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Embeddability between countable groups

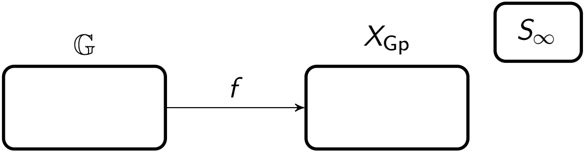
Proof *(sketch)*
J. Williams defined a Borel function

$$\begin{aligned} X_{Gr} &\longrightarrow X_{Gp} \\ T &\longmapsto G_T. \end{aligned}$$

Every G_T satisfies some small cancellation properties, which are used to prove that f is a reduction for both

- $\sqsubseteq_{\mathbb{G}} \leq_B \sqsubseteq_{\mathsf{Gp}}$,
- $=_{\mathbb{G}} \leq_B \cong_{\mathsf{Gp}}$.

Embeddability between countable groups



Let S_∞ be the Polish group of all permutations of $\mathbb{N}$.

$S_\infty \curvearrowright X_{\mathsf{Gp}}$ is continuous and $\cong_{\mathsf{Gp}}$ coincides with $E_{S_\infty}^{X_{\mathsf{Gp}}}$.

$$\begin{aligned}\Sigma(T) =& \{h \in S_\infty : j_{\mathsf{Gp}}(h, id \circ f(T)) = id \circ f(T)\} = \\ =& \{h \in S_\infty : h \in Aut(G_T)\}\end{aligned}$$

One can prove that $\Sigma : \mathbb{G} \to F(S_\infty)$ is Borel. □

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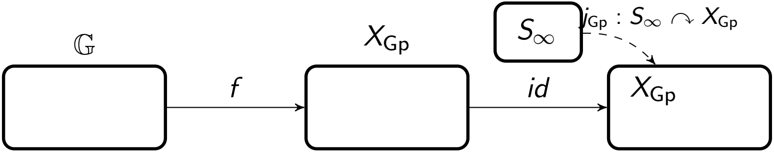

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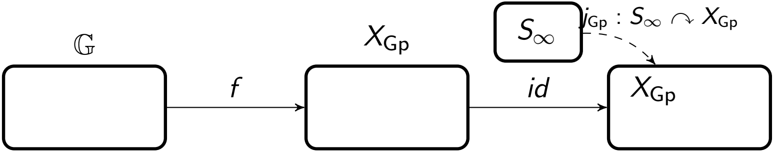

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Embeddability between Polish groups

Theorem (Ferenczi-Louveau-Rosendal 2009)

$\sqsubseteq_{\mathfrak{G}}$ *is* $\boldsymbol{\Sigma}^1_1$*-complete.*

Theorem (C.-Motto Ros)

$\sqsubseteq_{\mathfrak{G}}$ *is invariantly universal.*

By Uspenskij, every Polish group is homeomorphic to a closed subgroup of $\mathrm{Hom}([0,1]^{\mathbb{N}})$.
Let $\mathfrak{G} := F(\mathrm{Hom}([0,1]^{\mathbb{N}}))$ with the Effros Borel structure.

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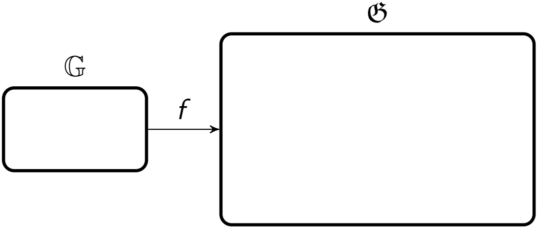
Proof *(sketch)*

By J. Williams, there exists a Borel function

$$\begin{aligned} X_{Gr} &\longrightarrow X_{Gp} \\ T &\longmapsto G_T \end{aligned}$$

witnessing $\sqsubseteq_{Gr} \leq_B \sqsubseteq_{Gp}$.

Embeddability between Polish groups



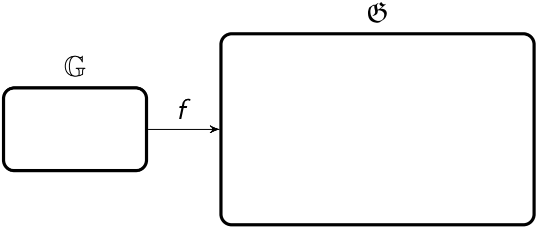
$$T \longmapsto (G_T, \mathcal{P}(G_T)) \rightsquigarrow \text{code of } (G_T, \mathcal{P}(G_T)) \text{ in } \mathfrak{G}$$

However, ran $f \subseteq D = \{F \in \mathfrak{G} : F \text{ is a discrete group}\}$.

Lemma

D is Π^1_1-complete.

Embeddability between Polish groups



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⚠ It is NOT possible to reduce $\cong_{\mathfrak{G}}$ to any Borel group action because $\cong_{\mathfrak{G}}$ is $\mathbf{\Sigma}^1_1$-complete.

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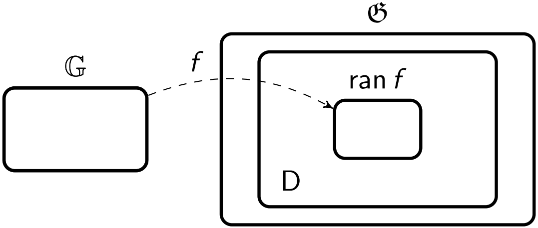

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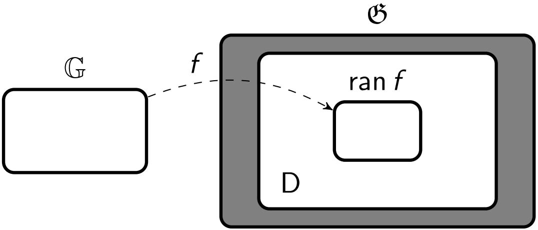


$\mathfrak{G} \setminus D$ is $\mathbf{\Sigma}^1_1$.

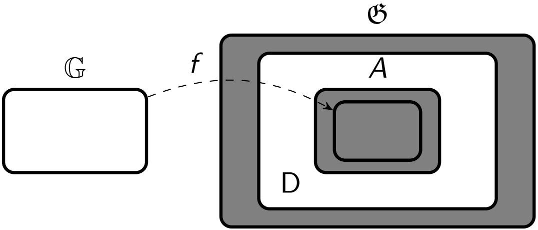
Let A be the $\cong_{\mathfrak{G}}$-saturation of ran f. That is,

$$A := \{F \in \mathfrak{G} : \exists T \in \mathbb{G}\, (F \cong_{\mathfrak{G}} f(T))\}.$$

A is $\mathbf{\Sigma}^1_1$. By the separation theorem for $\mathbf{\Sigma}^1_1$ equivalence relations, there is a Borel and $\cong_{\mathfrak{G}}$-invariant $B \subseteq \mathfrak{G}$ such that

$$B \supseteq A \quad \text{and} \quad B \cap (\mathfrak{G} \setminus D) = \varnothing.$$

Embeddability between Polish groups



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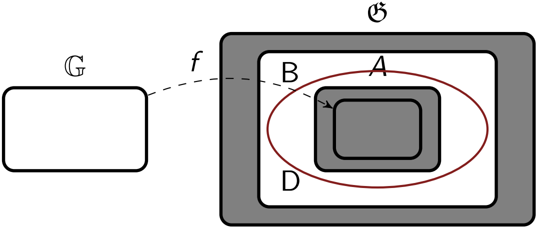

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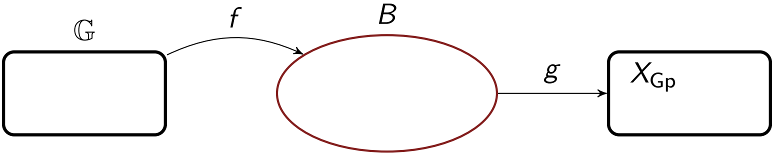

Every group in $B \subseteq D$ is a discrete Polish group, then we define a Borel $g : B \to X_{\mathrm{Gp}}$ such that $g(f(T)) \cong G_T$.

One can prove that he map

$$\begin{aligned}\Sigma : \mathbb{G} &\longrightarrow F(S_\infty)\\ T &\longmapsto \{h \in S_\infty : j_{\mathrm{Gp}}(h, g \circ f(T)) = g \circ f(T)\}\end{aligned}$$

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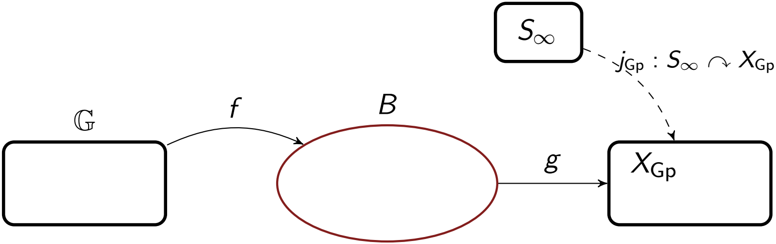

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