Expected values for the a.d. number and 3D-iterations

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This is a part of a joint work with V. Fischer, S. Friedman and D. Montoya-Amaya
Some basic notions

- For $f, g \in \omega^\omega$, $f \leq^* g$ (\textit{g dominates f}) means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$.

- The \textit{bounding number} $\mathfrak{b}$ is the least size of a $\leq^*$-unbounded subset of $\omega^\omega$.

- The \textit{dominating number} $\mathfrak{d}$ is the least size of a $\leq^*$-cofinal family in $\omega^\omega$. 
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- \( A \subseteq [\omega]^{\aleph_0} \) is an **a.d. (almost disjoint) family** if \( a \cap b \) is finite for any distinct \( a, b \in A \).
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- The a.d. number $\mathfrak{a}$ is the least size of an infinite mad (maximal a.d.) family.
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- For $f, g \in \omega^\omega$, $f \leq^* g$ (\emph{g dominates f}) means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$.

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- The \emph{dominating number} $\d$ is the least size of a $\leq^*$-cofinal family in $\omega^\omega$.

- $A \subseteq [\omega]^{\aleph_0}$ is an \emph{a.d. (almost disjoint) family} if $a \cap b$ is finite for any distinct $a, b \in A$.

- The \emph{a.d. number} $\alpha$ is the least size of an \emph{infinite mad (maximal a.d.) family}.

\textbf{Fact}

$b \leq \d$ and $b \leq \alpha$. 
Under CH, there is an infinite mad family which is preserved mad in any $C_\kappa$-extension. The above also works for any random algebra $B_\kappa$.

$(Brendle and Judah and Shelah 1993)$ $D$ adds a mad family of size $\aleph_1$.

$(Brendle 1995)$ $LOC$ and $E$ adds a mad family of size $\aleph_1$.

Theorem $(Steprans 1993)$ $C_{\omega_1}$ adds a mad family of size $\aleph_1$ which is preserved mad in further Cohen extensions.

Typically, it requires a lot of work to increase $a$ (beyond $b$).
Effect of some classical ccc posets on $\alpha$

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Find the value of $\alpha$ in generic extensions of some ccc FS-iteration,
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Conjecture
Many of these iterations force $a = b$... at least by technical changes in their construction.
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With $\mu = \aleph_1$, a FS-it. of length $\omega_1$ over a model with $c = \lambda$ works for (i) and forces $a = \aleph_1$. 
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(2) Construct the desired iteration afterwards while preserving the mad family.
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1. Add a mad family of certain size (expected to be $\mathfrak{b}$ at the end).
2. Construct the desired iteration afterwards while preserving the mad family.

**Theorem (Brendle and Fischer 2011)**

If $\kappa \leq \mu$ are uncountable regular and $\mu^{\aleph_0} = \mu$ then there is a ccc poset forcing $\mathfrak{b} = \mathfrak{a} = \kappa \leq \mathfrak{s} = \mathfrak{c} = \mu$. 
Adding a mad family

Definition (Hechler 1972)

For a set $X$ define the poset $\mathbb{H}_X$:

- **Conditions:**
  
  $F_p : \mathcal{P}_p \times n_p \to 2$ where $F_p \in [X]^{<\aleph_0}$ and $n_p < \omega$.

- **Order:**
  
  $q \leq p$ iff $p \subseteq q$ and, for any $i \in n_q \setminus n_p$, there is at most one $x \in F_p$ such that $p(x, i) = 1$. 
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Diagagram:

- $\omega$
- $n_q$
- $n_p$
- $F_p$
- $F_q$
- $X$
Adding a mad family

The poset adds generically a family $\mathcal{A}|X := \langle A_x : x \in X \rangle$ of subsets of $\omega$ where

$$i \in A_x \text{ iff } p(x, i) = 1 \text{ for some } p \in G.$$
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i \in A_x \text{ iff } p(x, i) = 1 \text{ for some } p \in G.
\]
$\mathcal{A}|X$ is a.d.. Additionally, when $X$ is uncountable, it is mad.

$\mathcal{H}_X$ is ccc, moreover, it has precaliber $\omega_1$. $X \subseteq Y$ implies $\mathcal{H}_X \preceq \mathcal{H}_Y$. If $C$ is a $\subseteq$-chain of sets and $Y = \bigcup C$ then $\mathcal{H}_Y = \limdir X \in C \mathcal{H}_X$.

Therefore, if $\delta$ is a limit ordinal, $\mathcal{H}_\delta$ comes from the FS-iteration $\langle \mathcal{H}_\alpha, \dot{Q}_\alpha \rangle$ $\alpha < \delta$ where $\dot{Q}_\alpha$ is $\sigma$-centered.

$\mathcal{H}_X \simeq C$ when $X$ is countable. $\mathcal{H}_X \simeq C_{\omega_1}$ when $|X| = \aleph_1$. 

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Definition (Brendle and Fischer 2011)

Let $M$ be a transitive model of ZFC, $\mathcal{A} = \{A_z : z \in \Omega\} \in M$ a family of infinite subsets of $\omega$ and $B^* \in [\omega]^{\aleph_0}$.

Lemma (Brendle and Fischer 2011)

In $V$, let $\Omega$ be a set and $z^* \in \Omega$. Then, in $V[H_{\Omega}]$, $A_{z^*}$ diagonalizes $V[H_{\Omega}] \setminus \{z^*\}$ outside $A|_{\Omega \setminus \{z^*\}}$. 
**Preservation properties**

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**Lemma (Brendle and Fischer 2011)**

Let $\mathcal{A}$, $M$ and $B^*$ as above. If $B^*$ diagonalizes $M$ outside $\mathcal{A}$ then

$|X \cap B^*| = \aleph_0$ for any $X \in M \cap [\omega]^{\aleph_0} \setminus \mathcal{I}(\mathcal{A})$. 
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**Lemma (Brendle and Fischer 2011)**

In $V$, let $\Omega$ be a set and $z^* \in \Omega$. Then, in $V^{H_\Omega}$, $A_{z^*}$ diagonalizes $V^{H_\Omega \setminus \{z^*\}}$ outside $\mathcal{A}|(\Omega \setminus \{z^*\})$. 
An application

**Theorem (essentially Brendle 1991)**

Let $\kappa \leq \mu$ be uncountable regular cardinals, $\mu \leq \lambda$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset forcing $\kappa \mu \lambda \aleph_1$ add($\mathcal{N}$) add($\mathcal{M}$) cov($\mathcal{M}$) non($\mathcal{N}$) and $\alpha = b = \kappa$.
An application

**Theorem**

Let $\kappa \leq \mu$ be uncountable regular cardinals, $\mu \leq \lambda$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset forcing

\[
\begin{array}{c}
\text{cov}(\mathcal{N}) & \text{non}(\mathcal{M}) & \text{cof}(\mathcal{M}) & \text{cof}(\mathcal{N}) \\
\text{add}(\mathcal{N}) & \text{add}(\mathcal{M}) & \text{cov}(\mathcal{M}) & \text{non}(\mathcal{N})
\end{array}
\]

and $\alpha = b = \kappa$. 
\[ \mathcal{A} := \mathcal{A}|_\kappa = \{ A_\beta : \beta < \kappa \} \text{ is a mad family in } V_{0,\kappa}. \]
\[ V^H_{\kappa} = V_{0,\kappa} \]
\[ V^H_{\beta+1} = V_{0,\beta+1} \]
\[ V^H_{\beta} = V_{0,\beta} \]
\[ V^H_1 = V_{0,1} \]
\[ V = V_{0,0} \]

\[ A := A|_\kappa = \{ A_\beta : \beta < \kappa \} \] is a mad family in \( V_{0,\kappa} \).

\( A_\beta \) diagonalizes \( V_{0,\beta} \) outside \( A|_\beta \) (for all \( \beta < \kappa \)).
\[ A := A\mid_\kappa = \{ A_\beta : \beta < \kappa \} \text{ is a mad family in } V_{0,\kappa}. \]

\[ A_\beta \text{ diagonalizes } V_{0,\beta} \text{ outside } A\mid_\beta \text{ (for all } \beta < \kappa). \]
$A := A|\kappa = \{ A_\beta : \beta < \kappa \}$ is a mad family in $V_{0,\kappa}$.

$A_\beta$ diagonalizes $V_{0,\beta}$ outside $A|\beta$ (for all $\beta < \kappa$).

$N_\alpha \in V_{\alpha+1,t(\alpha+1)}$ is a transitive model of ZFC of size $< \kappa$. 
\[ \mathcal{A} := \mathcal{A}|\kappa = \{ A_\beta : \beta < \kappa \} \text{ is a mad family in } V_{0,\kappa}. \]

\[ A_\beta \text{ diagonalizes } V_{0,\beta} \text{ outside } \mathcal{A}|\beta \text{ (for all } \beta < \kappa). \]

\[ N_\alpha \in V_{\alpha+1, t(\alpha+1)} \text{ is a transitive model of ZFC of size } < \kappa. \]

Parallel FS-iterations of length \( \lambda\mu \).
Fix $M \subseteq N$ transitive models of ZFC, $A \in M$ and $B^* \in N$ diagonalizing $M$ outside $A$.

$$B^* \in N \bullet$$

$$A \in M \bullet$$
Fix $M \subseteq N$ transitive models of ZFC, $A \in M$ and $B^* \in N$ diagonalizing $M$ outside $A$.

Lemma (Brendle and Fischer 2011)

Let $P \in M$ be a poset. Then, in $N^P$, $B^*$ diagonalizes $M^P$ outside $A$. 
Fix $M \subseteq N$ transitive models of ZFC, $A \in M$ and $B^* \in N$ diagonalizing $M$ outside $A$.

\[ B^* \in N \xrightarrow{\mathcal{E}^N} N^{\mathcal{E}^N} \]
\[ A \in M \xrightarrow{\mathcal{E}^M} M^{\mathcal{E}^M} \]

**Lemma (Brendle and Fischer 2011)**

Let $P \in M$ be a poset. Then, in $N^P$, $B^*$ diagonalizes $M^P$ outside $A$.

**Main Lemma**

In $N^{\mathcal{E}^N}$, $B^*$ diagonalizes $M^{\mathcal{E}^M}$ outside $A$. 
Fix $M \subseteq N$ transitive models of ZFC, $A \in M$ and $B^* \in N$ diagonalizing $M$ outside $A$.

\[\begin{align*}
B^* &\in N \xrightarrow{E^N} N^{E^N} \\
A &\in M \xrightarrow{E^M} M^{E^M}
\end{align*}\]

**Lemma (Brendle and Fischer 2011)**

Let $P \in M$ be a poset. Then, in $N^P$, $B^*$ diagonalizes $M^P$ outside $A$.

**Main Lemma**

In $N^{E^N}$, $B^*$ diagonalizes $M^{E^M}$ outside $A$.

The same holds for random forcing and Cohen forcing.
A general result

Theorem

Let $\kappa$ be an uncountable regular cardinal. After forcing with $\mathbb{H}_\kappa$, any further FS-iteration where each iterand is either

(i) in \{C, random, E\} or

(ii) a ccc poset of size $<\kappa$

preserves the mad family added by $\mathbb{H}_\kappa$. 
A general result

Theorem

Let $\kappa$ be an uncountable regular cardinal. After forcing with $H_\kappa$, any further FS-iteration where each iterand is either

(i) in \{\mathcal{C}, random, \mathcal{E}\} or

preserves the mad family added by $H_\kappa$. 
A general result

Theorem

Let $\kappa$ be an uncountable regular cardinal. After forcing with $\mathbb{H}_\kappa$, any further FS-iteration where each iterand is either

(i) in $\{C, \text{random}, E\}$ or
(ii) a ccc poset of size $< \kappa$,

preserves the mad family added by $\mathbb{H}_\kappa$. 
More examples

Corollary

Let $\theta_0 \leq \theta_1 \leq \kappa \leq \mu$ be uncountable regular cardinals, $\mu \leq \lambda$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset forcing

$$\text{add}(N) \quad \text{non}(M) \quad \text{cov}(M) \quad \text{cof}(N)$$

and $a = b = \kappa$. 
Corollary

Let $\theta_0 \leq \kappa \leq \mu$ be uncountable regular cardinals, $\mu \leq \lambda$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset forcing

and $\alpha = b = \kappa$. 
Corollary

Let $\theta_0 \leq \theta_1 \leq \kappa$ be uncountable regular cardinals, $\lambda$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset forcing

$$\begin{align*}
\theta_0 & \leq \theta_1 \\
\lambda & \leq \kappa
\end{align*}$$

and $\alpha = b = \kappa$. 
Theorem (M. 2013)

Let \( \mu \leq \nu \) be uncountable regular cardinals, \( \nu \leq \lambda \) such that \( \lambda^{\aleph_0} = \lambda \). Then, there is a ccc poset forcing...
... turned into a 3D-iteration!

**Theorem**

Let $\kappa \leq \mu \leq \nu$ be uncountable regular cardinals, $\nu \leq \lambda$ such that $\lambda^{\aleph_0} = \lambda$. Then, there is a ccc poset forcing

\[
\text{add}(N) \quad \text{non}(M) \quad \text{cov}(M) \quad \text{non}(N)
\]

and $\alpha = b$. 
$V^H_\infty = V_{0,0,\kappa}$
$V^{H_n} = V_{0,0,\kappa}$

Diagram with nodes labeled $V_{\alpha,0,\kappa}$, $V_{\alpha,0,\gamma+1}$, $V_{0,0,\gamma}$, $V_{0,0,1}$, $V_{0,0,0}$, and $A_0$, $A_\gamma$. Edges connect these nodes with labels $c_\alpha$ and $C$.
\[ V_{\alpha,0,\kappa} = V_{0,0,\kappa} \]

\[ V_{\alpha+1,0,\kappa} = V_{0,0,\gamma+1} \]

\[ V_{\alpha+1,0,\gamma+1} = V_{0,0,1} \]

\[ V_{\alpha+1,0,1} = V_{0,0,0} \]

\[ V_{\alpha+1,0,0} = V_{0,0} \]

\[ V_{\lambda_\nu,0,\kappa} = V_{0,0,\kappa+1} \]

\[ V_{\lambda_\nu,0,\gamma+1} = V_{0,0,1} \]

\[ V_{\lambda_\nu,0,1} = V_{0,0} \]

\[ V_{\lambda_\nu,0,0} = V_{0,0} \]

\[ V_{\lambda_\nu,0,\kappa+1} = V_{0,0,1} \]

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More examples

**Theorem**

Let $\theta_0 \leq \theta_1 \leq \kappa \leq \mu \leq \nu$ be uncountable regular cardinals, $\nu \leq \lambda$ such that $\lambda^{<\theta_1} = \lambda$. Then, there is a ccc poset forcing

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and $a = b$. 

What happens to $\alpha$ in a FS-iteration of $\mathcal{D}$ of length $\mu > \aleph_1$ (regular) over a ground model where $\mathfrak{c} > \mu$?
Questions

Question
What happens to $\alpha$ in a FS-iteration of $\mathbb{D}$ of length $\mu > \aleph_1$ (regular) over a ground model where $\kappa > \mu$?

Question
Is it possible to obtain $\alpha = b$ in Goldstern-M.-Shelah (2016) model of