These notes contain the complete proof of the following result:

**Theorem.** A Boolean $\sigma$-algebra $B$ is a measure algebra if and only if it is weakly distributive and uniformly concentrated.
A Boolean algebra is an algebra $B$ of subsets of a given nonempty set $S$, with Boolean operations $a \cup b$, $a \cap b$, $-a = S - a$, and the zero and unit elements $0 = \emptyset$ and $1 = S$.

A Boolean $\sigma$-algebra is a Boolean algebra $B$ such that every countable set $A \subset B$ has a supremum $\sup A = \bigvee A$ (and an infimum $\inf A = \bigwedge A$) in the partial ordering of $B$ by inclusion.

Let $B$ be a Boolean algebra and let $B^+ = B - \{0\}$. A set $A \subset B^+$ is an antichain if $a \cap b = 0$ whenever $a$ and $b$ are distinct elements of $A$. A partition $W$ (of $1$) is a maximal antichain, i.e. an antichain with $\bigvee W = 1$. $B$ satisfies the countable chain condition (ccc) if it has no uncountable antichains.
A measure (a strictly positive $\sigma$-additive probability measure) on a Boolean $\sigma$-algebra $B$ is a real valued function $m$ on $B$ such that

(i) $m(0) = 0$, $m(a) > 0$ for $a \neq 0$, and $m(1) = 1$
(ii) $m(a) \leq m(b)$ if $a \subset b$
(iii) $m(a \cup b) = m(a) + m(b)$ if $a \cap b = 0$
(iv) $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ if the $a_n$ are pairwise disjoint.

A measure algebra is a Boolean $\sigma$-algebra that carries a measure.

A function $m$ that satisfies (i), (ii) and (iii) above is called a (strictly positive) finitely additive measure. And $m$ satisfies (iv) if and only if it is continuous:

(iv) if $\{a_n\}_n$ is a decreasing sequence in $B$ with $\bigwedge_{n=1}^{\infty} a_n = 0$ then $\lim_n m(a_n) = 0$.

Every measure algebra is ccc because $B^+ = \bigcup_{n=1}^{\infty} \{C_n\}_n$ where $C_n = \{a : m(a) \geq 1/n\}$ and every antichain in $C_n$ has size $\leq n$. 

Thomas Jech
If \( \{a_n\}_n \) is a sequence in a Boolean \( \sigma \)-algebra \( B \), one defines

\[
\limsup_n a_n = \bigcap_{n=1}^{\infty} \bigvee_{k=n}^{\infty} a_k \quad \text{and} \quad \liminf_n a_n = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} a_k,
\]

and if \( \limsup_n a_n = \liminf_n a_n = a \), then \( a \) is the limit of the sequence, denoted \( \lim_n a_n \).

A sequence \( \{a_n\}_n \) converges to 0 if and only if \( \limsup_n a_n = 0 \) if and only if there exist \( b_n \geq a_n \), \( b_n \) decreasing, with \( \bigwedge_n b_n = 0 \). A sequence \( \{a_n\}_n \) converges to 1 if and only if \( \liminf_n a_n = 1 \).

(Exercise: If \( \lim a_n = \lim b_n = 0 \) then \( \lim(a_n \cup b_n) = 0 \).)

If \( \{a_n\}_n \) is an antichain then \( \lim a_n = 0 \).

In a measure algebra, if \( \lim a_n = 0 \) then \( \lim m(a_n) = 0 \). If \( \sum m(a_n) < \infty \) then \( \lim a_n = 0 \).
An algebraic characterization of measure algebras

A Boolean $\sigma$-algebra $B$ is weakly distributive if whenever $\{W_n\}_n$ is a sequence of countable maximal antichains then each $W_n$ has a finite subset $E_n$ such that $\lim_n \bigcup E_n = 1$.

Equivalently, if for every $k$, $\{a_{kn}\}_n$ is an increasing sequence with $\bigvee_n a_{kn} = 1$ then there exists a function $f : N \to N$ such that $\lim a_{k,f(k)} = 1$.

Or, if for every $k$, $\{a_{kn}\}_n$ is a decreasing sequence with $\bigwedge_n a_{kn} = 0$ then there exists a function $f : N \to N$ such that $\lim a_{k,f(k)} = 0$.

A Boolean $\sigma$-algebra $B$ is uniformly weakly distributive if there exists a sequence of functions $\{F_n\}_n$ such that for each countable maximal antichain $W$, $F_n(W)$ is a finite subset of $W$, and if $\{W_n\}_n$ is a sequence of countable maximal antichains then $\lim_n \bigcup F_n(W_n) = 1$.

If $B$ is a measure algebra then $B$ is uniformly weakly distributive: For every $n$, if $W$ is a countable maximal antichain, let $F_n(W) = E \subseteq W$ be such that $m(\bigcup E) \geq 1 - 1/2^n$. 
A Boolean $\sigma$-algebra $B$ is *concentrated* if for every sequence $\{A_n\}$ of finite antichains with $|A_n| \geq 2^n$ there exist $a_n \in A_n$ such that $\lim a_n = 0$.

$B$ is *uniformly concentrated* if there exists a function $F$ such that for each finite antichain $A$, $F(A)$ is an element of $A$, and if $A_n$ is a sequence of finite antichains with $|A_n| \geq 2^n$ then $\lim_n F(A_n) = 0$.

A measure algebra is uniformly concentrated: if $A$ is a finite antichain, let $F(A)$ be an element of $A$ of least measure (then $m(F(A)) \leq 1/|A|$.)

**Theorem.** A Boolean $\sigma$-algebra $B$ is a measure algebra if and only if it is weakly distributive and uniformly concentrated.
A fragmentation of $B$ is a sequence of subsets $C_1 \subset C_2 \subset \ldots \subset C_n \subset \ldots$ such that $\bigcup_n C_n = B^+$ and for every $n$, if $a \in C_n$ and $a \leq b$ then $b \in C_n$.

A fragmentation is $\sigma$-finite cc if for every $n$, every antichain $A \subset C_n$ is finite.

A fragmentation is $\sigma$-bounded cc if for every $n$ there is a constant $K_n$ such that every antichain $A \subset C_n$ has size $\leq K_n$.

A fragmentation is $G_\delta$ if for every $n$, no sequence in $C_n$ converges to 0.

A fragmentation is tight if whenever $\{a_n\}_n$ is a sequence such that $a_n \notin C_n$ for every $n$, then $\lim a_n = 0$.

A fragmentation is graded if for every $n$, whenever $a \cup b \in C_n$ then either $a \in C_{n+1}$ or $b \in C_{n+1}$.
The name $G_\delta$ comes from the fact that if $B$ is weakly distributive and the fragmentation is $G_\delta$ then the set $\{0\}$ is a $G_\delta$ set in the convergence topology.

A $G_\delta$ fragmentation is $\sigma$-finite cc.

In a measure algebra, the fragmentation defined by $C_n = \{a : m(a) \geq 1/2^n\}$ has all the above properties.

**Lemma.** (a) If $B$ is uniformly concentrated then $B$ has a tight $\sigma$-bounded cc fragmentation.
(b) If $B$ is weakly distributive and has a tight $\sigma$-finite cc fragmentation then $B$ is uniformly weakly distributive.

Consequently, a weakly distributive, uniformly concentrated $\sigma$-algebra is uniformly weakly distributive.

**Proof.** (a) Let $F$ be a function on finite antichains that witnesses that $B$ is uniformly concentrated. For every $n$ let $C_n$ be the set of all $a \neq 0$ such that there is no antichain $A$ with $|A| \geq 2^n$ and $a \leq F(A)$. 

---

Thomas Jech  
Measure Algebras
It is easy to see that $C_n$ is a fragmentation: If $a \notin \bigcup_n C_n$ then for every $n$ there exists an antichain $A_n$ with $|A_n| \geq 2^n$ and $a \leq F(A_n)$. Hence $a \leq \lim F(A_n) = 0$, and so $a = 0$.

The fragmentation is tight because if $a_n \notin C_n$ then $a_n \leq F(A_n)$ for some $A_n$ and so $\lim a_n = 0$. It is $\sigma$-bounded cc because if $A \subset C_n$ is an antichain then $F(A) \in C_n$ and so $|A| < 2^n$.

(b) Let $B$ be weakly distributive and let $\{C_n\}_n$ be a tight $\sigma$-finite cc fragmentation; we shall find the functions $F_n$ witnessing that $B$ is uniformly weakly distributive.

Let $n \in N$, and let $W$ be a countable maximal antichain. We claim that there exists a finite set $E \subset W$ such that for every finite $E' \subset W - E$, $\bigcup E' \notin C_n$: otherwise we find an infinite sequence $\{E_k\}_k$ of pairwise disjoint nonempty subsets of $W$ with $\bigcup E_k \in C_n$, an infinite antichain in $C_n$. We let $F_n(W)$ be this $E$. 
Now let $W_n$, $n \in \mathbb{N}$, be countable maximal antichains. We claim that $\lim \bigcup F_n(W_n) = 1$. Since $B$ is weakly distributive, there exist finite sets $E_n \subset W_n$ such that $\lim \bigcup E_n = 1$. For each $n$ let $a_n = \bigcup E_n - \bigcup F_n(W_n)$. By the claim above, $a_n \notin C_n$. Because $\{C_n\}_n$ is tight, we have $\lim a_n = 0$, and because $\bigcup F_n(W_n) \geq \bigcup E_n \cap (-a_n)$, we have $\lim \bigcup F_n(W_n) = \lim \bigcup E_n = \lim -a_n = 1$.

**Lemma.** A Boolean $\sigma$–algebra is uniformly weakly distributive if and only if it has a tight $G_\delta$ fragmentation.

**Proof.** First assume that $B$ is uniformly weakly distributive and let $F_n$ be functions witnessing it. For each $n$ we let $C_n$ be the set of all $a$ such that for some $k \leq n$ $a \cap \bigcup F_k(W) \neq 0$ for every countable maximal antichain $W$. 
To show that $\{C_n\}_n$ is a fragmentation, we show that 
$\bigcup_n C_n = B^+$: if $a \notin C_n$ for all $n$ then for all $k$ there is a $W_k$ such 
that $a \cap \bigcup F_k(W_k) = 0$, and because $\lim(a \cap \bigcup F_k(W_k) = a$ we 
have $a = 0$.

To show that $\{C_n\}_n$ is tight, let $a_n \notin C_n$ for each $n$. For each $n$ 
there is a $W_n$ such that $a_n \cap b_n = 0$ where $b_n = \bigcup F_n(W_n)$. Since 
$\lim b_n = 1$, we have $\lim -b_n = 0$, and because $a_n \leq -b_n$, it follows 
that $\lim a_n = 0$.

To show that $\{C_n\}_n$ is $G_\delta$, let $n \in N$ and let $\lim a_k = 0$; it suffices 
to find a $k \in N$ such that $a_k \notin C_n$. We may assume that $\{a_k\}_k$ is 
strictly decreasing and let $W$ be the maximal antichain 
$\{a_{k-1} - a_k\}_k$ where $a_0 = 1$. Let $E = F_1(W) \cup \ldots \cup F_n(W)$. There 
exists a $k$ large enough so that $a_k \cap \bigcup E = 0$. It follows that 
$a_k \notin C_n$. 

For the converse, let \( \{C_n\}_n \) be a tight \( G_\delta \) fragmentation. In view of the preceding lemma it suffices to show that \( B \) is weakly distributive. For every \( k \), let \( \{a_{kn}\}_n \) be a decreasing sequence with \( \bigwedge_n a_{kn} = 0 \). We shall find a function \( f : N \to N \) such that \( \lim a_{k,f(k)} = 0 \). Given \( k \in N \), there is some \( f(k) \) such that \( a_{k,f(k)} \notin C_k \), as \( \{C_n\}_n \) is \( G_\delta \). Because \( \{C_n\}_n \) is tight, \( \lim a_{k,f(k)} = 0 \) follows.

A tight \( G_\delta \) fragmentation is essentially unique: if \( C_n \) and \( C'_n \) are such, then for each \( n \) there is a \( k \) such that \( C_n \subset C'_k \). (If not, there exists an \( n \) such that for all \( k \) there is some \( a_k \in C_n - C'_k \) and so \( \lim a_k = 0 \)).

(In the appendix we show that \( B \) is uniformly weakly distributive if and only if \( B \) is a Maharam algebra.)

**Lemma.** If \( \{C_n\}_n \) is a \( G_\delta \) fragmentation of \( B \) and if \( B \) is concentrated then \( \{C_n\}_n \) is \( \sigma \)-bounded cc.
Proof. By contradiction, assume that for some $n$, there exist arbitrarily large finite antichains in $C_n$, and for each $k$, let $A_k$ be an antichain in $C_n$ of size $\geq 2^k$. Since $B$ is concentrated, there exist $a_k \in A_k$ with $\lim a_k = 0$, a contradiction.

Lemma. If $\{C_n\}_n$ is a tight $G_\delta$ fragmentation then for every $n$ there exists a $k > n$ such that for every $c \in C_n$, if $c = a \cup b$ then either $a \in C_k$ or $b \in C_k$.

Thus if $B$ has a tight $G_\delta$ fragmentation then $B$ has one that is also graded (i.e. $a \cup b \in C_n$ implies that either $a \in C_{n+1}$ or $b \in C_{n+1}$.)

Proof. Otherwise, for every $k > n$ there exist $c_k = a_k \cup b_k \in C_n$ such that $a_k \notin C_k$ and $b_k \notin C_k$. By tightness, $\lim a_k = \lim b_k = 0$ and so $\lim c_k = 0$, a contradiction.

In conclusion, we proved that a weakly distributive uniformly concentrated Boolean $\sigma$–algebra has a graded $\sigma$–bounded $cc$ fragmentation.
Kelley’s Theorem

In this lecture we introduce Kelley’s condition for the existence of finitely additive measure on a Boolean algebra. But first we show how the measure problem reduces to finitely additive measures.

**Theorem** (Pinsker, Kelley). A Boolean $\sigma$-algebra $B$ carries a measure if and only if it is weakly distributive and carries a finitely additive measure.

**Proof.** Let $m$ be a finitely additive measure on $B$. We let

$$\mu(b) = \inf \{ \lim_n m(b \cap u_n) \}$$

where the infimum is taken over all increasing sequences $\{u_n\}_n$ with $\bigvee_n u_n = 1$.

We show that $\mu$ is a $\sigma$-additive measure, and if $B$ is weakly distributive then $\mu$ is strictly positive.
Kelley’s Theorem

First show that \( \mu(a \cup b) = \mu(a) + \mu(b) \) if \( a \cap b = \emptyset \). If \( s = \{u_n\}_n \), let \( \mu_s(x) = \lim_n m(x \cap u_n) \). Clearly, \( \mu_s(a \cup b) = \mu_s(a) + \mu_s(b) \), and so \( \mu(a) + \mu(b) \leq \mu_s(a \cup b) \). Hence \( \mu(a) + \mu(b) \leq \mu(a \cup b) \).

For each \( \varepsilon > 0 \) there is a sequence \( s = s_a = \{u_n\}_n \) such that \( \mu_s(a) \leq \mu(a) + \varepsilon \), and similarly \( s_b = \{v_n\}_n \). Let \( s = \{u_n \cap v_n\}_n \).

Then \( \mu(a \cup b) \leq \mu_s(a \cup b) = \mu_s(a) + \mu_s(b) \leq \mu_{s_a}(a) + \mu_{s_b}(b) \leq \mu(a) + \mu(b) + 2\varepsilon \), and the equality follows.

To show the continuity of \( \mu \), let \( a_n \) be a decreasing sequence with \( \bigwedge_n a_n = \emptyset \); we show that \( \lim \mu(a_n) = 0 \). Let \( \varepsilon > 0 \).

Let \( M = \lim m(a_n) \), and let \( K \) be such that \( m(a_K) - M < \varepsilon \). Let \( s = \{u_n\}_n \) where \( u_n = -a_n \). As for all \( k, n \geq K \), \( m(a_k - a_n) < \varepsilon \), we have, for every \( k \geq K \),

\[ \mu_s(a_k) = \lim_n m(a_k \cap u_n) = \lim_n m(a_k - a_n) \leq \varepsilon, \]

and hence \( \mu(a_k) \leq \varepsilon \).
Finally, assume that $B$ is weakly distributive, and let $b \in B$ be such that $\mu(b) = 0$; we show that $b = 0$. As $\mu(b) = 0$, there is for each $k$ an increasing sequence $\{u_{kn}\}_n$ with $\bigvee_n u_{kn} = 1$ such that $\lim_n m(b \cap u_{kn}) < 1/k$.

By weak distributivity there is a function $f$ such that $\lim u_{k,f(k)} = 1$. Hence $\bigvee_n \bigwedge_{k \geq n} u_{k,f(k)} = 1$. For each $k$ let $a_k = b \cap u_{k,f(k)}$. We have $\bigvee_n \bigwedge_{k \geq n} a_k = b$. But because $m(a_k) < 1/k$, it follows that $\bigwedge_{k \geq n} a_k = 0$, and so $b = \bigvee_n \bigwedge_{k \geq n} a_k = 0$. 
The intersection number

In order to state and prove Kelley’s Theorem, we now work with Boolean set algebras, and use the term “finitely additive” for measures that are not necessarily strictly positive - when we need the condition $m(a) > 0$ if $a \neq 0$ we call $m$ strictly positive.

Let $B$ be a Boolean set algebra, $B \subset P(S)$ for some set $S$. Let $C$ be a subset of $B^+$. For every finite sequence $s = \langle c_1, \ldots, c_n \rangle$ in $C$, let $\kappa_s = k/n$ where $k$ is the largest size of a subset $J \subset \{1, \ldots, n\}$ such that $\bigcap_{i \in J} c_i$ is nonempty. The intersection number of $C$ is the infimum $\kappa = \inf \kappa_s$ over all finite sequences $s$ in $C$.

(The sequences $s$ do not have to be nonrepeating.)

Note that for any $n_0$, the infimum $\inf \kappa_s$ taken over all sequences $s$ of length $n \geq n_0$ is still $\kappa$: if $s$ is a sequence of length $n < n_0$, let $t$ be such that $t \cdot n \geq n_0$, and let $s^*$ be a sequence we get when repeating each term of $s$ $t$-times. Then $\kappa_{s^*} = \kappa_s$. 
The intersection number

To better understand the significance of the intersection number, assume that $m$ is a finitely additive measure on $B$, and let $C \subset B^+$. Let $M$ be such that $m(c) \geq M$ for all $c \in C$. We show that the intersection number $\kappa$ of $C$ is at least $M$.

Let $s = \langle c_1, \ldots, c_n \rangle$ be a sequence in $C$. For each $i \leq n$, let $K_i$ be the characteristic function of $c_i$, i.e. $K_i(x) = 1$ if $x \in c_i$ and $= 0$ if $x \notin c_i$. Let $g = \sum_{i=1}^{n} K_i$ and consider $l_s = \int g \text{ } dm$, the area below the graph of $g$. Since $\int K_i = m(c_i)$, we have $l_s = \sum_i m(c_i) \geq M \cdot n$ and so (because $m(S) = 1$) $\|g\| = \max_{x \in S} g(x) \geq l_s \geq M \cdot n$.

Thus there exists some $x \in S$ such that $\sum_i K_i(x) \geq M \cdot n$; in other words, $x$ belongs to at least $M \cdot n$ members of the sequence. Hence $\kappa_s \geq M \cdot n/n = M$ and consequently $\kappa \geq M$. 

Thomas Jech  
Measure Algebras
Kelley’s Theorem

**Theorem** (Kelley). Let $C \subset B^+$ have a positive intersection number $\kappa$. Then there exists a finitely additive measure $m$ on $B$ such that $m(c) \geq \kappa$ for all $c \in C$.

**Corollary.** If a Boolean algebra $B$ has a fragmentation $\{C_n\}$ such that each $C_n$ has a positive intersection number, then $B$ carries a strictly positive finitely additive measure.

**Proof of Corollary.** For each $n$ let $m_n$ be positive on $C_n$. If we let $m(a) = \sum_n m_n(a)/2^n$, $m$ is a strictly positive, finitely additive measure on $B$.

To prove Kelley’s Theorem and construct a finitely additive measure on $B$ we shall consider the vector space of all bounded functions on $S$ (including all characteristic functions $K_a$ for all $a \in B$) and find a linear functional $F$ such that $F(1) = 1$, $F(K_a) \geq 0$, and $F(K_c) \geq \kappa$ for all $c \in C$. Then we let $m(a) = F(K_a)$ for all $a \in B$. 
Kelley’s Theorem

To find the linear functional we use the Hahn-Banach Theorem:

Let \( p \) be a function such that \( p(x) \geq 0 \) for all \( x \),
\[
p(x + y) \leq p(x) + p(y), \quad p(\alpha x) = \alpha p(x) \text{ for all } \alpha \geq 0, \text{ and}
\]
\[
p(1) \geq 1.
\]
Then there exists a linear functional \( F \) such that \( F(1) = 1 \) and
\[
F(x) \leq p(x) \text{ for all } x.
\]

**Proof of Kelley’s Theorem.** Let \( C \subset B^+ \subset P(S) \) and let \( \kappa \) be
the intersection number of \( C \). Let \( V \) be the vector space of all
bounded functions on \( S \) with the supremum norm
\[
||f|| = \sup\{|f(x)| : x \in S\}.
\]
We shall find a linear functional \( F \) on \( V \) such that \( 0 \leq F(K_a) \leq 1 \) for all \( a \in B \), \( F(1) = 1 \) and
\[
F(K_c) \geq \kappa \text{ for all } c \in C.
\]
Consider the convex hull of the set \( \{K_c : c \in C\} \):
\[
G = \left\{ \sum_{i=1}^{i=m} \alpha_i K_{c_i} : c_i \in C, \ 0 \leq \alpha_i \leq 1, \ \sum \alpha_i = 1 \right\}
\]
**Lemma.** For every $g \in G$, $\|g\| \geq \kappa$.

**Proof.**
First consider rational coefficients $\alpha_i$: for each $i \leq m$, $\alpha_i = l_i/n$ with $\sum_i l_i = n$, and $g(x) = f(x)/n$ where $f = \sum_i l_i \cdot K_{c_i}$. Consider the sequence $s$ in $C$ of length $n$ where each $c_i$ is repeated $l_i$ times. By definition of $\kappa$ there are $k$ terms of $s$ with nonempty intersection such that $k/n \geq \kappa$. Let $x$ be a point in the intersection; it follows that $f(x) \geq k$. Hence $g(x) \geq k/n \geq \kappa$.

For arbitrary $\alpha_i$ let $\varepsilon > 0$. There are rational approximations $\beta_i$ of the $\alpha_i$ such that $\kappa \leq \| \sum \beta_i K_{c_i} \| \leq \| \sum \alpha_i K_{c_i} \| + \varepsilon$. Hence $\|g\| \geq \kappa - \varepsilon$, and so $\|g\| \geq \kappa$. 
Kelley’s Theorem

Let \( Q = \{ \alpha(g - \kappa) + f : g \in G, \alpha \geq 0, f \geq 0 \} \).

The set \( Q \) contains all \( K_b, b \in B \), (because \( K_b \geq 0 \)) and is convex: if \( f \) and \( g \) are in \( Q \) and \( \alpha + \beta = 1 \) (\( \alpha, \beta > 0 \)) then \( \alpha f + \beta g \in Q \).

Clearly, it suffices to verify this for \( f \) and \( g \) in \( G \) and that is easy.

Let \( \delta = 1 - \kappa \), and let \( U \) be the open ball \( \{ h : \|h\| < \delta \} \) of radius \( \delta \). Using the vector space convention \( A - B = \{ a - b : a \in A, b \in B \} \), consider the set

\[ U - Q = \{ h - \alpha(g - \kappa) - f : \|h\| < \delta, g \in G, \alpha \geq 0, f \geq 0 \}. \]

The set \( U - Q \) is convex, and because \( 0 \in Q \), we have \( U \subset U - Q \), and so for every \( v \in V \) there exists a positive number \( \alpha \) such that \( \alpha v \in U \subset U - Q \). Now define

\[ p(v) = \inf \{ \gamma > 0 : \frac{v}{\gamma} \in U - Q \}. \]

If \( \gamma = \alpha + \beta \) and \( \frac{x}{\alpha}, \frac{y}{\beta} \in U - Q \) then \( \frac{x+y}{\gamma} = \frac{\alpha}{\gamma} \left( \frac{x}{\alpha} \right) + \frac{\beta}{\gamma} \left( \frac{y}{\beta} \right) \in U - Q \) by convexity, and so \( p(x + y) \leq p(x) + p(y) \).

Clearly, \( p(\alpha v) = \alpha p(v) \) for all \( \alpha \geq 0 \).
Kelley’s Theorem

Finally, let $\gamma \leq 1$; we show that $1/\gamma \notin U - Q$, hence $p(1) \geq 1$. If $1/\gamma \in U - Q$ then $1/\gamma = h - f$ (and $h = f + 1/\gamma$) where $h \in U$ and $f \in Q$. Since $f \in Q$, there exists, by the Lemma, some $x \in S$ such that $f(x) \geq 0$, and so $f(x) + 1/\gamma \geq 1/\gamma \geq 1$. But $h(x) < \delta \leq 1$.

Now we apply the Hahn-Banach Theorem to this function $p$. Note that for all $f \in Q$, $-f \in U - Q$ and hence $p(-f) \leq 1$.

There exists a linear functional $F$ such that $F(1) = 1$ and $F(x) \leq p(x)$ for all $x \in V$. If $f \in Q$, then $F(-f) \leq p(-f) \leq 1$ and so $F(f) \geq -1$. As this is true for all $f \in Q$ and $Q$ is closed under multiples by all $\alpha \geq 0$, it must be the case that $F(f) \geq 0$ for all $f \in Q$. In particular, $F(K_b) \geq 0$ for all $b \in B$.

Also, if $g \in G$, then $g - \kappa \in Q$ and hence $F(g - \kappa) \geq 0$, i.e. $F(g) \geq \kappa$. Consequently, $F(K_c) \geq \kappa$ for all $c \in C$.

When we let $m(b) = F(K_b)$ for all $b \in B$, $m$ is a finitely additive measure on $B$ with $m(c) \geq \kappa$ for all $c \in C$. 
The Kalton-Roberts Proof

We complete the proof by showing that Kelley’s condition applies:

**Theorem.** Let $B$ be a Boolean algebra that has a graded $\sigma$-bounded cc fragmentation $\{C_n\}$. Then for every $n$, $C_n$ has a positive intersection number.

To prove the theorem, we adapt the Kalton-Roberts proof, an ingenious combinatorial argument that converts finite bounds for the size of antichains into positive intersection numbers.

**Lemma.** Let $M$ and $P$ be finite sets with $|M| = m$ and $|P| = p \leq m$, and let $k, 3 \leq k \leq p$ be an integer such that $p/k \geq 15 \cdot m/p$. Then there exists an indexed family $\{A_i : i \in M\}$ such that each $A_i$ is a three point subset of $P$ and such that for every $I \subset M$ with $|I| \leq k$,

$$|\bigcup_{i \in I} A_i| > |I|.$$
It follows that for every $I \subset M$ with $|I| \leq k$ there exists a one-to-one choice function $f_I$ on \{A$_i$ : $i \in I$\}.

The last statement of the lemma follows by Hall’s “Marriage Theorem”:

A family \{A$_1$, ..., A$_n$\} of finite sets has a set of distinct representatives if and only if $|\bigcup_{i \in I} A_i| \geq |I|$ for every $I \subset \{1, ..., n\}$.

(For a proof of Hall’s Theorem, see Appendix.)
The proof of the lemma uses a counting argument:

**Proof.** Consider the families \( \{A_i : i \in M\} \) of three point subsets of \( P \). Let us call such a family *bad* if \( |\bigcup_{i \in I} A_i| \leq |I| \) for some \( I \subset M \), \( |I| \leq k \). If a family is bad then for some \( n, 3 \leq n \leq k \), there exist sets \( I \subset M \) and \( J \subset P \), \( |I| = |J| = n \) such that \( A_i \subset J \) for every \( i \in I \).

There are \( \binom{p}{3} \) three-point subsets of \( P \) and \( \binom{n}{3} \) three-point subsets of \( J \). Of the \( \binom{p}{3}^n \) families \( \{A_i\}_{i \in I} \) with domain \( I \), \( \binom{n}{3}^n \) are such that \( \bigcup_{i \in I} A_i \subset J \). The ratio of such families (for \( 3 \leq n \leq p \)) is \((\frac{n}{3}/\binom{p}{3})^n \leq (n^3/p^3)^n = n^{3n}/p^{3n}\) because \( \frac{n}{3}/\binom{p}{3} \leq n^3/p^3 \).

Because there are \( \binom{m}{n} \) subsets \( I \subset M \) of size \( n \) and \( \binom{p}{n} \) subsets \( J \subset P \) of size \( n \), the probability that a family \( \{A_i\}_{i \in M} \) is bad is at most

\[
\Pi = \sum_{n=3}^{n=k} \binom{m}{n} \binom{p}{n} \frac{n^{3n}}{p^{3n}}.
\]
The Kalton-Roberts Proof

We have \((\binom{m}{n}) \cdot (\binom{p}{n}) \cdot \frac{n^3}{p^3} \leq \frac{m^n}{n!} \cdot \frac{p^n}{n!} \cdot \frac{n^3}{p^3}\). Using \(e^x > \frac{x^n}{n!}\) we get \(e^n n! > n^n\), hence \(\frac{1}{n!} < \frac{e^n}{n^n}\), and so

\[
\frac{m^n}{n!} \cdot \frac{p^n}{n!} \cdot \frac{n^3}{p^3} < \frac{e^{2n} n^m n^{n^m}}{p^{2n}} = \left(\frac{e^2 \cdot \frac{n}{p} \cdot \frac{m}{p}}{p}\right)^n.
\]

For \(n \leq k\) we have \(e^2 \cdot \frac{n}{p} \cdot \frac{m}{p} \leq e^2 \cdot \frac{k}{p} \cdot \frac{m}{p} \leq \frac{e^2}{15} < \frac{1}{2}\) because we assumed \(\frac{p}{k} \geq 15 \frac{m}{p}\) and because \(2e^2 < 15\). Therefore

\[
\Pi < \sum_{n=3}^{n=k} \left(\frac{1}{2}\right)^n < 1.
\]

Consequently, there exists a family \(\{A_i : i \in M\}\) that is not bad, and so \(|\bigcup_{i \in I} > |I|\) for every \(I \subset M\) of size \(\leq k\).
The Kalton-Roberts Proof

We shall now apply the Kalton-Roberts method:

**Proof of Theorem.** Let \( \{C_n\} \) be a graded \( \sigma \)-bounded cc fragmentation of a Boolean algebra \( B \), and let us fix an integer \( n \). We prove that the intersection number of each \( C_n \) is positive, namely \( \geq 1/(30K^2) \) where \( K = K_{n+2} \) is the maximal size of an antichain in \( C_{n+2} \).

We show that for every \( m \geq 100K^2 \), and every sequence \( \{c_1, ..., c_m\} \) in \( C_n \) there exists some \( J \subset m \) of size \( \geq m/(30K^2) \) such that \( \bigcap_{i \in J} c_i \) is nonempty.

Let \( M = \{1, ..., m\} \) with \( m \geq 100K^2 \) and let \( c_1, ..., c_m \in C_n \). For each \( I \subset M \), let

\[
b_I = \bigcap\{c_i : i \in I\} \cap \bigcap\{-c_i : i \notin I\}.
\]

The sets \( b_I \) are pairwise disjoint (some may be empty) and \( \bigcup\{b_I : I \subset M\} = 1 \). Note that for each \( i \in M \), \( \bigcup\{b_I : i \in I\} = c_i \).

We shall find a sufficiently large set \( J \subset M \) with nonempty \( b_J \).
We shall apply the Lemma. First let \( k \geq 3 \) be the largest \( k \) such that \( k/m < 1/(30K^2) \) (there is such because \( 3/m \leq 3/(100K^2) \)). We have \( k < m \) and \((k + 1)/m \geq 1/(30K^2)\). Then let \( p \) be the largest \( p \geq k \) such that \( p/m < 1/K \) (there is such because \( k/m < 1/K \)).

We verify the assumption of the lemma, \( p/k \geq 15m/p \) (using \( p/(p + 1) \geq 3/4 \)):

\[
\frac{p}{k} = \frac{p}{p + 1} \cdot \frac{p + 1}{m} \cdot \frac{m}{k} \geq \frac{3}{4} \cdot \frac{1}{K} \cdot 30K^2 \geq 20K
\]

and

\[
15 \frac{m}{p} \leq 15 \cdot \frac{p + 1}{p} \cdot \frac{m}{p + 1} \leq 15 \cdot \frac{4}{3} \cdot K = 20K.
\]
Now we apply the Lemma: Let $P = \{1,\ldots,p\}$. There exist three point sets $A_i \subset P$, $i \in M$, and one-to-one functions $f_I$ on all $I \subset M$ of size $\leq k$ with $f_I(i) \in A_i$ for all $i \in I$.

We shall prove that there exists a $J \subset M$ of size $\geq k + 1$ (and hence $\geq m/(30K^2)$) such that $b_J$ is nonempty. By contradiction, assume that there is no such $J$. Then

$$\bigcup\{b_I : |I| \leq k\} = 1$$

and for each $i \in M$, $c_i = \bigcup\{b_I : |I| \leq k \text{ and } i \in I\}$.

For each $i \in M$ and $j \in P$ let

$$a_{ij} = \bigcup\{b_I : |I| \leq k, i \in I \text{ and } f_I(i) = j\}.$$

Note that for each $i \in M$, $c_i = a_{i,j_1} \cup a_{i,j_2} \cup a_{i,j_3}$ where $A_i = \{j_1, j_2, j_3\}$. 
Let $j \in P$. We claim that the $a_{ij}, i \in M$, are pairwise disjoint: If $a_{i_1,j} \cap a_{i_2,j}$ is nonempty, then because the $b_I$ are pairwise disjoint there is some $I$ such that $i_1 \in I$ and $i_2 \in I$, and because $f_I(i_1) = j = f_I(i_2)$ and $f_I$ is one-to-one, we have $i_1 = i_2$. Hence the $a_{ij}, i \in M$, are pairwise disjoint, and so only at most $K$ of them belong to $C_{n+2}$.

Consequently, at most $p \cdot K$ of the $a_{ij}$ belong to $C_{n+2}$ and because $pK < m$, there exists an $i$ such that $a_{ij} \notin C_{n+2}$ for all (three) $j \in A_i$.

But then $c_i = a_{i,j_1} \cup a_{i,j_2} \cup a_{i,j_3} \notin C_n$ because the fragmentation is graded. This contradicts the assumption that $c_i \in C_n$. 